

Some topics related to metrics and norms, including ultrametrics and ultranorms

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Abstract

Here we look at some geometric properties related to connectedness and topological dimension 0, especially in connection with norms on vector spaces over fields with absolute value functions, which may be non-archimedean.

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Part I

Metrics and norms

1 q -Metrics

Let M be a set, and let q be a positive real number. A nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ is said to be a q -metric on M if it satisfies the following three conditions. First,

$$(1.1) \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y.$$

Second,

$$(1.2) \quad d(x, y) = d(y, x) \quad \text{for every } x, y \in M.$$

Third,

$$(1.3) \quad d(x, z)^q \leq d(x, y)^q + d(y, z)^q \quad \text{for every } x, y, z \in M.$$

Of course, (1.3) is the version of the triangle inequality associated to q . If this holds with $q = 1$, then we simply say that $d(x, y)$ is a *metric* on M . Thus $d(x, y)$ is a q -metric on M if and only if $d(x, y)^q$ is a metric on M .

Similarly, a nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ is said to be an *ultrametric* on M if it satisfies (1.1), (1.2), and

$$(1.4) \quad d(x, z) \leq \max(d(x, y), d(y, z)) \quad \text{for every } x, y, z \in M,$$

instead of (1.3). Clearly, for each $q > 0$, (1.4) is equivalent to asking that

$$(1.5) \quad d(x, z)^q \leq \max(d(x, y)^q, d(y, z)^q) \quad \text{for every } x, y, z \in M.$$

If $d(x, y)$ is an ultrametric on M , then it follows that $d(x, y)$ is a q -metric on M for every $q > 0$, since (1.5) implies (1.3). In this case, we also get that $d(x, y)^q$ is an ultrametric on M for every $q > 0$. Note that the discrete metric on any set M is an ultrametric, which is defined by putting $d(x, y)$ equal to 1 when $x \neq y$, and to 0 when $x = y$.

It is sometimes convenient to reformulate (1.3) as saying that

$$(1.6) \quad d(x, z) \leq (d(x, y)^q + d(y, z)^q)^{1/q} \quad \text{for every } x, y, z \in M.$$

Observe that

$$(1.7) \quad \max(a, b) \leq (a^q + b^q)^{1/q} \leq 2^{1/q} \max(a, b)$$

for any pair a, b of nonnegative real numbers, which implies that

$$(1.8) \quad \lim_{q \rightarrow \infty} (a^q + b^q)^{1/q} = \max(a, b).$$

Thus (1.4) corresponds to taking the limit as $q \rightarrow \infty$ in (1.6), so that one might think of an ultrametric as being a q -metric with $q = \infty$.

If $0 < q_1 < q_2 < \infty$, then

$$(1.9) \quad a^{q_2} + b^{q_2} \leq \max(a, b)^{q_2 - q_1} (a^{q_1} + b^{q_1})$$

for every $a, b \geq 0$. We also have that

$$(1.10) \quad \max(a, b) \leq (a^{q_1} + b^{q_1})^{1/q_1},$$

as in (1.7), so that

$$(1.11) \quad a^{q_2} + b^{q_2} \leq (a^{q_1} + b^{q_1})^{((q_2 - q_1)/q_1) + 1} = (a^{q_1} + b^{q_1})^{q_2/q_1}$$

for every $a, b \geq 0$. Equivalently,

$$(1.12) \quad (a^{q_2} + b^{q_2})^{1/q_2} \leq (a^{q_1} + b^{q_1})^{1/q_1}$$

for every $a, b \geq 0$ when $0 < q_1 < q_2 < \infty$. If $d(x, y)$ is a q_2 -metric on M for some $q_2 > 0$, then it follows that $d(x, y)$ is a q_1 -metric on M as well when $0 < q_1 < q_2$. Of course, the topology on M determined by the metric $d(x, y)^{q_1}$ is the same as the topology on M determined by the metric $d(x, y)^{q_2}$ in this case.

2 Open and closed balls

Let M be a set, and suppose that $d(x, y)$ is a q -metric on M for some positive real number q . If $x \in M$ and r is a positive real number, then the open ball centered at x with radius r is defined as usual by

$$(2.1) \quad B(x, r) = \{z \in M : d(x, z) < r\}.$$

Equivalently,

$$(2.2) \quad B(x, r) = \{z \in M : d(x, z)^q < r^q\},$$

which is the open ball in M centered at x with radius r^q with respect to the metric $d(\cdot, \cdot)^q$. If $y \in B(x, r)$, so that $d(x, y)^q < r^q$, then let t be the positive real number determined by

$$(2.3) \quad t^q = r^q - d(x, y)^q.$$

It is easy to see that

$$(2.4) \quad B(y, t) \subseteq B(x, r)$$

under these conditions, because $d(\cdot, \cdot)^q$ is a metric on M .

Let us say that $U \subseteq M$ is an open set in M if for each $x \in U$ there is an $r > 0$ such that

$$(2.5) \quad B(x, r) \subseteq U.$$

This is analogous to the standard definition for metric spaces, and it is equivalent to saying that U is an open set in M with respect to the metric $d(\cdot, \cdot)^q$, because of (2.2). In particular, this defines a topology on M , which is the same as the

topology on M determined by the metric $d(\cdot, \cdot)^q$. Open balls in M are open sets with respect to this topology, by (2.4), which corresponds to the standard argument for metric spaces.

Similarly, if $d(\cdot, \cdot)$ is an ultrametric on M , then

$$(2.6) \quad B(y, r) \subseteq B(x, r)$$

for every $y \in B(x, r)$, which is the same as (2.4) with $t = r$. More precisely, this holds when $d(x, y) < r$, which is symmetric in x and y . Thus we also have that

$$(2.7) \quad B(x, r) \subseteq B(y, r)$$

when $d(x, y) < r$, so that

$$(2.8) \quad B(x, r) = B(y, r)$$

in this situation.

Let $d(\cdot, \cdot)$ be a q -metric on M for some $q > 0$ again. The closed ball in M centered at $x \in M$ with radius $r \geq 0$ with respect to $d(\cdot, \cdot)$ is defined by

$$(2.9) \quad \overline{B}(x, r) = \{z \in M : d(x, z) \leq r\}.$$

As before, this is the same as

$$(2.10) \quad \overline{B}(x, r) = \{z \in M : d(x, z)^q \leq r^q\},$$

which is the closed ball in M centered at x with radius r^q with respect to the metric $d(\cdot, \cdot)^q$. If $y \in \overline{B}(x, r)$, so that $d(x, y)^q \leq r^q$, then let t be the nonnegative real number determined by (2.3). In analogy with (2.4), we have that

$$(2.11) \quad \overline{B}(y, t) \subseteq \overline{B}(x, r),$$

since $d(\cdot, \cdot)^q$ is a metric on M . If $d(\cdot, \cdot)$ is an ultrametric on M , then

$$(2.12) \quad \overline{B}(y, r) \subseteq \overline{B}(x, r)$$

for every $y \in \overline{B}(x, r)$, which is the same as (2.11) with $t = r$. Equivalently, (2.12) holds when $d(x, y) \leq r$, which is symmetric in x and y . Thus the opposite inclusion also holds in this case, so that

$$(2.13) \quad \overline{B}(x, r) = \overline{B}(y, r)$$

for every $x, y \in M$ with $d(x, y) \leq r$. This implies that closed balls in M are open sets when $d(\cdot, \cdot)$ is an ultrametric on M .

3 Some related facts

If $d(x, y)$ is a q -metric on a set M for some positive real number q , then we can reexpress (1.3) as

$$(3.1) \quad d(x, z)^q - d(y, z)^q \leq d(x, y)^q \quad \text{for every } x, y, z \in M.$$

Of course, this is nontrivial only when $d(y, z)^q < d(x, z)^q$, which is to say that

$$(3.2) \quad d(y, z) < d(x, z).$$

If $d(\cdot, \cdot)$ is an ultrametric on M , then (1.4) and (3.2) imply that

$$(3.3) \quad d(x, z) \leq d(x, y).$$

In this case, we also have that

$$(3.4) \quad d(x, y) \leq \max(d(x, z), d(z, y)) \leq d(x, z)$$

when $d(y, z) \leq d(x, z)$. It follows that

$$(3.5) \quad d(x, y) = d(x, z)$$

when $d(\cdot, \cdot)$ is an ultrametric on M and $x, y, z \in M$ satisfy (3.2), by combining (3.3) and (3.4).

Let $d(\cdot, \cdot)$ be a q -metric on M for some $q > 0$ again, and put

$$(3.6) \quad V(x, r) = \{z \in M : d(x, z) > r\} = \{z \in M : d(x, z)^q > r^q\}$$

for every $x \in M$ and $r \geq 0$, which is the same as the complement of $\overline{B}(x, r)$ in M . If $z \in V(x, r)$, then let t be the positive real number determined by

$$(3.7) \quad t^q = d(x, z)^q - r^q.$$

If $y \in B(z, t)$, so that $d(y, z)^q < t^q$, then (3.1) implies that $d(x, y)^q > r^q$, which means that $y \in V(x, r)$. This shows that

$$(3.8) \quad B(z, t) \subseteq V(x, r),$$

which implies that $V(x, r)$ is an open set in M , and hence that $\overline{B}(x, r)$ is a closed set in M . If $d(\cdot, \cdot)$ is an ultrametric on M , then (3.8) holds with $t = d(x, z)$, because of (3.5).

Let $d(\cdot, \cdot)$ be any q -metric on M again, and put

$$(3.9) \quad W(x, r) = \{z \in M : d(x, z) \geq r\} = \{z \in M : d(x, z)^q \geq r^q\}$$

for every $x \in M$ and $r > 0$, which is the same as the complement of $B(x, r)$ in M . If $z \in W(x, r)$, then let t be the nonnegative real number determined by (3.7). If $y \in \overline{B}(z, t)$, so that $d(y, z)^q \leq t^q$, then (3.1) implies that $d(x, y)^q \geq r^q$, and thus $y \in W(x, r)$. It follows that

$$(3.10) \quad \overline{B}(z, t) \subseteq W(x, r)$$

under these conditions, which is trivial when $d(x, z) = r$, so that $t = 0$. Note that $W(x, r)$ is a closed set in M for every $x \in M$ and $r > 0$, since it is the complement of an open set.

If $d(\cdot, \cdot)$ is an ultrametric on M , then

$$(3.11) \quad B(z, d(x, z)) \subseteq W(x, r)$$

for every $z \in W(x, r)$. More precisely, if $y \in B(z, d(x, z))$, then (3.2) holds, which implies that (3.5) holds as well. If we also have $z \in W(x, r)$, then it follows that

$$(3.12) \quad d(x, y) = d(x, z) \geq r$$

for every $y \in B(z, d(x, z))$, so that $y \in W(x, r)$, as desired. In particular, this shows that $W(x, r)$ is an open set in M for every $x \in M$ and $r > 0$ when $d(\cdot, \cdot)$ is an ultrametric on M . Thus $B(x, r)$ is a closed set in M for every $x \in M$ and $r > 0$ in this case.

4 Absolute value functions

Let k be a field, and let q be a positive real number again. A nonnegative real-valued function $|\cdot|$ defined on k is said to be a *q-absolute value function* if it satisfies the following three conditions. First, for each $x \in k$,

$$(4.1) \quad |x| = 0 \quad \text{if and only if} \quad x = 0.$$

Second,

$$(4.2) \quad |xy| = |x| |y| \quad \text{for every } x, y \in k.$$

Third,

$$(4.3) \quad |x + y|^q \leq |x|^q + |y|^q \quad \text{for every } x, y \in k.$$

If (4.3) holds with $q = 1$, then we may simply say that $|\cdot|$ is an *absolute value function* on k . Equivalently, $|x|$ is a q -absolute value function on k if and only if $|x|^q$ is an absolute value function on k .

Suppose for the moment that $|\cdot|$ is a nonnegative real-valued function on k that satisfies (4.1) and (4.2). Let us use 1 to denote the multiplicative identity element in k , as well as usual positive integer, depending on the context. Thus $|1| > 0$, by (4.1), since $1 \neq 0$ in k , by definition of a field. We also have that

$$(4.4) \quad |1| = |1^2| = |1|^2,$$

which implies that

$$(4.5) \quad |1| = 1.$$

Similarly, if $x \in k$ satisfies $x^n = 1$ for some positive integer n , then we get that

$$(4.6) \quad |x^n| = |x|^n = 1,$$

and hence

$$(4.7) \quad |x| = 1.$$

Let $-x$ be the additive inverse of $x \in k$, which is equal to $(-1)x$, where -1 is the additive inverse of 1 in k . In particular, $(-1)^2 = 1$ in k , which implies that

$$(4.8) \quad |-1| = 1,$$

as before. It follows that

$$(4.9) \quad |-x| = x$$

for every $x \in k$. If $|\cdot|$ is a q -absolute value function on k , then we get that

$$(4.10) \quad d(x, y) = |x - y|$$

defines a q -metric on k . More precisely, this uses (4.9) to get the symmetry condition (1.2).

A nonnegative real-valued function $|\cdot|$ on k is said to be an *ultrametric absolute value function* on k if

$$(4.11) \quad |x + y| \leq \max(|x|, |y|) \quad \text{for every } x, y \in k.$$

This implies that (4.10) defines an ultrametric on k . As before, for each $q > 0$, (4.11) is equivalent to asking that

$$(4.12) \quad |x + y|^q \leq \max(|x|^q, |y|^q) \quad \text{for every } x, y \in k.$$

If $|\cdot|$ is an ultrametric absolute value function on k , then it follows that $|\cdot|$ is a q -absolute value function on k for every $q > 0$, because (4.12) implies (4.3). We also get that $|x|^q$ is an ultrametric absolute value function on k for every $q > 0$ in this case.

As in Section 1, we can reformulate (4.3) as saying that

$$(4.13) \quad |x + y| \leq (|x|^q + |y|^q)^{1/q} \quad \text{for every } x, y \in k.$$

Using (1.8), (4.11) corresponds to taking the limit as $q \rightarrow \infty$ in (4.13), so that an ultrametric absolute value function may be considered as a q -absolute value function with $q = \infty$. We have also seen that the right side of the inequality in (4.13) decreases monotonically in q , by (1.12). If $0 < q_1 < q_2 < \infty$, and if $|\cdot|$ is a q_2 -absolute value function on k , then it follows that $|\cdot|$ is a q_1 -absolute value function on k too.

The *trivial absolute value function* may be defined on any field k by putting $|x| = 1$ when $x \neq 0$, and $|0| = 0$. It is easy to see that this defines an ultrametric absolute value function on k , for which the corresponding ultrametric (4.10) is the same as the discrete metric. Suppose for the moment that $|\cdot|$ is a nonnegative real-valued function on a field k that satisfies (4.1) and (4.2), and which is not the trivial absolute value function on k . This means that there is an $x \in k$ such that $x \neq 0$ and $|x| \neq 1$, and we may as well suppose that

$$(4.14) \quad 0 < |x| < 1,$$

since otherwise we can replace x with $1/x$. Of course, we also have that

$$(4.15) \quad |1/x| = 1/|x| > 1$$

in this case.

It is well known that the standard absolute value functions on the fields \mathbf{R} , \mathbf{C} of real and complex numbers are absolute value functions in the sense described in this section. Hence they are also q -absolute value functions when $0 < q < 1$, as before. However, it is easy to see that they are not q -absolute value functions when $q > 1$, even when restricted to the field \mathbf{Q} of rational numbers.

5 Some additional properties

Let k be a field, and let \mathbf{Z}_+ be the set of positive integers. If $x \in k$ and $n \in \mathbf{Z}_+$, then let $n \cdot x$ be the sum of n x 's in k . Note that

$$(5.1) \quad n_1 \cdot (n_2 \cdot x) = (n_1 n_2) \cdot x$$

for every $x \in k$ and $n_1, n_2 \in \mathbf{Z}_+$, and

$$(5.2) \quad n \cdot (xy) = (n \cdot x)y = x(n \cdot y)$$

for every $x, y \in k$ and $n \in \mathbf{Z}_+$. In particular,

$$(5.3) \quad n^j \cdot 1 = (n \cdot 1)^j$$

for every $j, n \in \mathbf{Z}_+$.

An absolute value function $|\cdot|$ on k is said to be *archimedean* if there are positive integers n such that $|n \cdot 1|$ is as large as one wants. Equivalently, $|\cdot|$ is archimedean when

$$(5.4) \quad |n \cdot 1| > 1$$

for some $n \in \mathbf{Z}_+$, since this implies that

$$(5.5) \quad |n^j \cdot 1| = |n \cdot 1|^j \rightarrow \infty \quad \text{as } j \rightarrow \infty,$$

by (5.3). Otherwise, $|\cdot|$ is *non-archimedean* when

$$(5.6) \quad |n \cdot 1| \leq 1$$

for every $n \in \mathbf{Z}_+$. The previous argument shows that it is enough to check that $|n \cdot 1|$ is bounded for $n \in \mathbf{Z}_+$, to get that $|\cdot|$ is non-archimedean. It is easy to see that ultrametric absolute value functions are non-archimedean, using the ultrametric version (4.11) of the triangle inequality. Conversely, it can be shown that non-archimedean absolute value functions satisfy the ultrametric version of the triangle inequality, as in Lemma 1.5 on p16 of [1], and Theorem 2.2.2 on p28 of [5]. There is an analogous statement for a q -absolute value function $|\cdot|$ on k for any $q > 0$, which can be derived from the previous statement for absolute value functions applied to $|x|^q$.

A pair of absolute value functions $|\cdot|_1, |\cdot|_2$ on k are said to be *equivalent* if there is a positive real number a such that

$$(5.7) \quad |x|_2 = |x|_1^a$$

for every $x \in k$. This implies that the topologies on k determined by the metrics associated to $|\cdot|_1, |\cdot|_2$ as in (4.10) are the same. Conversely, if the topologies on k determined by the metrics associated to $|\cdot|_1$ and $|\cdot|_2$ are the same, then one can show that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k , as in Lemma 3.2 on p20 of [1], and Lemma 3.1.2 on p42 of [5]. Similarly, if $|\cdot|_1$ and $|\cdot|$ are q_1 and q_2 -absolute value functions on k for some $q_1, q_2 > 0$, then let us say that $|\cdot|_1$ and $|\cdot|_2$ are equivalent when (5.7) holds for some $a > 0$. This is the same as saying that

$$(5.8) \quad |x|_2^{q_2} = (|x|_1^{q_1})^{a q_2 / q_1}$$

for every $x \in k$, so that $|x|_1^{q_1}$ and $|x|_2^{q_2}$ are equivalent as absolute value functions on k .

Let $|\cdot|$ be an absolute value function on k , which leads to a metric on k as in (4.10), and hence a topology on k . Using standard arguments, one can check that addition and multiplication on k are continuous as mappings from $k \times k$ into k , where $k \times k$ is equipped with the corresponding product topology. Similarly,

$$(5.9) \quad x \mapsto 1/x$$

is continuous as a mapping from $k \setminus \{0\}$ into itself.

If k is not already complete as a metric space with respect to the metric associated to $|\cdot|$, then one can obtain a completion of k in the usual way. The field operations on k can be extended to the completion in a natural way, so that the completion of k is also a field. The absolute value function on k can be extended to an absolute value function on the completion of k as well, in such a way that the metric associated to the extension of the absolute value function to the completion of k is the same as the metric already given on the completion of k . If $|\cdot|$ is an ultrametric absolute value function on k , then the extension of $|\cdot|$ to the completion of k is an ultrametric absolute value function too. Of course, k is a dense subset of its completion, by construction.

Let $|\cdot|$ be an ultrametric absolute value function on any field k . If $x, y \in k$ and $|x - y| \leq |x|$, then

$$(5.10) \quad |y| \leq \max(|x|, |x - y|) \leq |x|.$$

If $|x - y| < |x|$, then

$$(5.11) \quad |x| \leq \max(|y|, |x - y|)$$

implies that $|x| \leq |y|$. Combining this with (5.10), we get that

$$(5.12) \quad |x| = |y|$$

when $|x - y| < |x|$.

If $x \in k$ and n is a nonnegative integer, then

$$(5.13) \quad (1 - x) \sum_{j=0}^n x^j = 1 - x^{n+1},$$

where x^j is interpreted as being equal to 1 when $j = 0$, as usual. It follows that

$$(5.14) \quad \sum_{j=0}^n x^j = \frac{1 - x^{n+1}}{1 - x}$$

for every $n \geq 0$ when $x \neq 1$. Let $|\cdot|$ be an absolute value function on k , and suppose that $|x| < 1$, so that

$$(5.15) \quad |x^{n+1}| = |x|^{n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies that

$$(5.16) \quad \lim_{n \rightarrow \infty} \sum_{j=0}^n x^j = \frac{1}{1 - x}$$

when $|x| < 1$, where the limit is taken with respect to the metric associated to $|\cdot|$ on k .

6 p -Adic numbers

If p is a prime number, then the p -adic absolute value $|x|_p$ of a rational number x is defined as follows. Of course, $|0|_p = 0$. Otherwise, if $x \neq 0$, then x can be expressed as

$$(6.1) \quad x = p^j (a/b),$$

where a , b , and j are integers, $a, b \neq 0$, and neither a nor b is divisible by p . In this case, we put

$$(6.2) \quad |x|_p = p^{-j}.$$

One can check that this defines an ultrametric absolute value function on \mathbf{Q} , so that the corresponding p -adic metric

$$(6.3) \quad d_p(x, y) = |x - y|_p$$

is an ultrametric on \mathbf{Q} .

The field \mathbf{Q}_p of p -adic numbers is obtained by completing \mathbf{Q} with respect to the p -adic metric, as in the previous section. The natural extension of the p -adic absolute value function to \mathbf{Q}_p is also called the p -adic absolute value, and denoted $|\cdot|_p$. Similarly, the natural extension of the p -adic metric to \mathbf{Q}_p is called the p -adic metric too, and is denoted $d_p(\cdot, \cdot)$. By construction, these extensions of the p -adic absolute value and metric to \mathbf{Q}_p are related as in (6.3). Note that $|\cdot|_p$ is an ultrametric absolute value function on \mathbf{Q}_p , and that $d_p(\cdot, \cdot)$ is an ultrametric on \mathbf{Q}_p , because of the corresponding properties on \mathbf{Q} . The possible values of the p -adic absolute value and metric on \mathbf{Q}_p are 0 and integer powers of p , as on \mathbf{Q} . This can be obtained from the construction of the completion, or from the fact that \mathbf{Q} is dense in \mathbf{Q}_p .

The set of p -adic integers is defined by

$$(6.4) \quad \mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \leq 1\},$$

which is a closed set in \mathbf{Q}_p with respect to the topology determined by the p -adic metric. Note that the set \mathbf{Z} of ordinary integers is contained in \mathbf{Z}_p , by definition of the p -adic metric on \mathbf{Q} . It follows that the closure of \mathbf{Z} in \mathbf{Q}_p is contained in \mathbf{Z}_p , and in fact \mathbf{Z}_p is equal to the closure of \mathbf{Z} in \mathbf{Q}_p . To see this, let $y \in \mathbf{Z}_p$ be given, and remember that y can be approximated by elements of \mathbf{Q} with respect to the p -adic metric, since \mathbf{Q} is dense in \mathbf{Q}_p . If $w \in \mathbf{Q}$ satisfies $|y - w|_p \leq 1$, then $|w|_p \leq 1$, by the ultrametric version of the triangle inequality. This implies that w can be expressed as a/b for some $a, b \in \mathbf{Z}$, where $b \neq 0$ and b is not divisible by p , by the definition of the p -adic absolute value on \mathbf{Q} . Because the integers modulo p form a field, there is a $c \in \mathbf{Z}$ such that $bc = 1 - pz$ for some $z \in \mathbf{Z}$. Thus

$$(6.5) \quad w = \frac{a}{b} = \frac{ac}{bc} = \frac{ac}{1 - pz}.$$

Of course, $|pz|_p = (1/p)|z|_p \leq 1/p < 1$, and so we can apply (5.16) with $x = pz$ to get that

$$(6.6) \quad w = \lim_{n \rightarrow \infty} ac \sum_{j=0}^n p^j z^j,$$

where the limit is taken with respect to the p -adic metric. This shows that w can be approximated by integers with respect to the p -adic metric when $w \in \mathbf{Q}$ and $|w|_p \leq 1$. It follows that every $y \in \mathbf{Z}_p$ can be approximated by integers with respect to the p -adic metric, since y can be approximated by $w \in \mathbf{Q}$ with $|w|_p \leq 1$, as before.

It is easy to see that \mathbf{Z}_p is a subgroup of \mathbf{Q}_p with respect to addition, because of the ultrametric version of the triangle inequality. Similarly,

$$(6.7) \quad p^j \mathbf{Z}_p = \{p^j x : x \in \mathbf{Z}_p\} = \{y \in \mathbf{Q}_p : |y| \leq p^{-j}\}$$

is a subgroup of \mathbf{Q}_p with respect to addition for every $j \in \mathbf{Z}$. One can also check that \mathbf{Z}_p is a subring of \mathbf{Q}_p , and that $p^j \mathbf{Z}_p$ is an ideal in \mathbf{Z}_p when $j \geq 0$.

Thus the quotient

$$(6.8) \quad \mathbf{Z}_p / p^j \mathbf{Z}_p$$

is defined as a commutative ring for every nonnegative integer j . The natural inclusion of \mathbf{Z} into \mathbf{Z}_p may be considered as a ring homomorphism, which leads to a ring homomorphism from \mathbf{Z} into (6.8), by composition with the quotient homomorphism from \mathbf{Z}_p onto (6.8). The kernel of this homomorphism from \mathbf{Z} into (6.8) is equal to

$$(6.9) \quad \mathbf{Z} \cap (p^j \mathbf{Z}_p) = p^j \mathbf{Z},$$

using the definition of the p -adic absolute value on \mathbf{Z} in the second step. Hence the homomorphism from \mathbf{Z} into (6.8) leads to an injective ring homomorphism from

$$(6.10) \quad \mathbf{Z} / p^j \mathbf{Z}$$

into (6.8). The usual homomorphism from \mathbf{Z} into (6.8) is actually surjective, because \mathbf{Z} is dense in \mathbf{Z}_p with respect to the p -adic metric. This implies that the

we get an ring isomorphism from (6.10) onto (6.8) for each $j \geq 0$. In particular, (6.8) has exactly p^j elements for each $j \geq 0$.

It follows that for each nonnegative integer j , \mathbf{Z}_p can be expressed as the union of p^j pairwise-disjoint translates of $p^j \mathbf{Z}_p$. Of course, the translates of $p^j \mathbf{Z}_p$ in \mathbf{Q}_p are the same as closed balls of radius p^{-j} with respect to the p -adic metric. This implies that \mathbf{Z}_p is totally bounded in \mathbf{Q}_p , since \mathbf{Z}_p can be covered by finitely many ball of arbitrarily small radius. It is well known that a subset of a complete metric space is compact if and only if it is closed and totally bounded. This shows that \mathbf{Z}_p is compact in \mathbf{Q}_p , because \mathbf{Z}_p is closed and totally bounded in \mathbf{Q}_p , and \mathbf{Q}_p is complete by construction.

An analogous argument implies that $p^l \mathbf{Z}_p$ is compact in \mathbf{Q}_p for every integer l . This can also be obtained from the compactness of \mathbf{Z}_p and continuity of multiplication on \mathbf{Q}_p . Similarly, one can use continuity of translations on \mathbf{Q}_p to get that every closed ball in \mathbf{Q}_p is compact. It follows that closed and bounded subsets of \mathbf{Q}_p are compact, since closed subsets of compact sets are compact. More precisely, it suffices to use the compactness of $p^l \mathbf{Z}_p$ for each $l \in \mathbf{Z}$, because every bounded subset of \mathbf{Q}_p is contained in $p^l \mathbf{Z}_p$ for some l .

7 q -Norms

Let k be a field, and let V be vector space over k . Also let $|\cdot|$ be a q -absolute value function on k for some positive real number q . A nonnegative real-valued function N on V is said to be a q -norm on V if it satisfies the following three conditions. First, for every $v \in V$,

$$(7.1) \quad N(v) = 0 \quad \text{if and only if} \quad v = 0.$$

Second,

$$(7.2) \quad N(tv) = |t| N(v) \quad \text{for every } t \in k \text{ and } v \in V.$$

Third,

$$(7.3) \quad N(v+w)^q \leq N(v)^q + N(w)^q \quad \text{for every } v, w \in V.$$

If $q = 1$, then we may simply say that N is a *norm* on V .

Remember that $|\cdot|$ is a q -absolute value function on k if and only if $|x|^q$ is an absolute value function on k . In this case, $N(v)$ is a q -norm on V with respect to $|x|$ on k if and only if $N(v)^q$ is a norm on V with respect to $|x|^q$ on k .

As usual, (7.3) can be reformulated as saying that

$$(7.4) \quad N(v+w) \leq (N(v)^q + N(w)^q)^{1/q} \quad \text{for every } v, w \in V.$$

We have seen that the right side of this inequality decreases monotonically in q , as in (1.12). If $0 < q_1 < q_2 < \infty$ and $|\cdot|$ is a q_2 -absolute value function on k , then $|\cdot|$ is a q_1 -absolute value function on k too, as in Section 4. If we suppose in addition that N is a q_2 -norm on V , then it follows that N is a q_1 -norm on V as well.

Suppose for the moment that $|\cdot|$ is a nonnegative real-valued function on k , and that N is a nonnegative real-valued function on V that satisfies (7.1),

(7.2), and (7.3) for some $q > 0$. If $V \neq \{0\}$, then one can check that $|\cdot|$ has to be a q -absolute value function on k under these conditions. Of course, if $|\cdot|$ is a q -absolute value function on k , then $|\cdot|$ may also be considered as a q -norm on k , where k is considered as a one-dimensional vector space over itself.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on k . A nonnegative real-valued function N on V is said to be an *ultranorm* if it satisfies (7.1), (7.2), and

$$(7.5) \quad N(v + w) \leq \max(N(v), N(w)) \quad \text{for every } v, w \in V.$$

As usual, for each $q > 0$, (7.5) is equivalent to asking that

$$(7.6) \quad N(v + w)^q \leq \max(N(v)^q, N(w)^q) \quad \text{for every } v, w \in V.$$

If N is an ultranorm on V , then it follows that N is a q -norm on V for every $q > 0$, because (7.6) implies (7.3). This also uses the fact that $|\cdot|$ is a q -absolute value function on k for every $q > 0$ when $|\cdot|$ is an ultrametric absolute value function on k , as in Section 4.

Similarly, if $|x|$ is an ultrametric absolute value function on k , then $|x|^q$ is an ultrametric absolute value function on k for every $q > 0$, as in Section 4. If $N(v)$ is an ultranorm on V with respect to $|x|$ on k , then $N(v)^q$ is an ultranorm on V with respect to $|x|^q$ on k for every $q > 0$, by (7.6).

Suppose for the moment again that $|\cdot|$ is a nonnegative real-valued function on k , and that N is a nonnegative real-valued function on V that satisfies (7.1), (7.2), and (7.5). If $V \neq \{0\}$, then one can check that $|\cdot|$ has to be an ultrametric absolute value function on k , as before. If $|\cdot|$ is an ultrametric absolute value function on k , then $|\cdot|$ may also be considered as an ultranorm on k , as a one-dimensional vector space over itself.

As in previous situations, (7.5) corresponds to taking the limit as $q \rightarrow \infty$ in (7.3), because of (1.8). Thus an ultranorm may be considered as a q -norm with $q = \infty$.

If $|\cdot|$ is a q -absolute value function on k , and if N is a q -norm on V with respect to $|\cdot|$, then

$$(7.7) \quad d(v, w) = N(v - w)$$

defines a q -metric on V . Similarly, if $|\cdot|$ is an ultrametric absolute value function on k , and if N is an ultranorm on V , then (7.7) is an ultrametric on V .

Consider the function N defined on V by $N(v) = 1$ when $v \neq 0$, and $N(0) = 0$. This is an ultranorm on V with respect to the trivial absolute value function on k , which is known as the *trivial ultranorm* on V . The ultrametric on V associated to the trivial ultranorm as in (7.7) is the same as the discrete metric on V .

8 Supremum metrics and norms

Let X and M be nonempty sets, and let $d(\cdot, \cdot)$ be a q -metric on M for some $q > 0$. As usual, a subset of M is said to be *bounded* with respect to $d(\cdot, \cdot)$ if it

is contained in a ball of finite radius in M . Similarly, a function f on X with values in M is said to be bounded if $f(X)$ is a bounded set in M . Let $B(X, M)$ be the space of bounded functions on X with values in M . If $f, g \in B(X, M)$, then $d(f(x), g(x))$ is a bounded nonnegative real-valued function on X , so that

$$(8.1) \quad \sup_{x \in X} d(f(x), g(x))$$

is defined as a nonnegative real number. It is easy to see that (8.1) defines a q -metric on $B(X, M)$, which may be described as the *supremum q -metric*. If $d(\cdot, \cdot)$ is an ultrametric on M , then (8.1) is an ultrametric on $B(X, M)$, which corresponds to the previous statement with $q = \infty$.

One can define Cauchy sequences in M with respect to $d(\cdot, \cdot)$ in the same way as for a metric. If every Cauchy sequence of elements of M converges to an element of M with respect to the topology determined by $d(\cdot, \cdot)$, then we say that M is complete with respect to $d(\cdot, \cdot)$, as usual. Any positive power of $d(\cdot, \cdot)$ determines the same collection of Cauchy sequences in M , and leads to an equivalent version of completeness. In particular, this permits one to reduce to the case of ordinary metrics, using suitable powers of $d(\cdot, \cdot)$. If M is complete with respect to $d(\cdot, \cdot)$, then $B(X, M)$ is complete with respect to (8.1), by standard arguments.

Suppose for the moment that X is a topological space, and let $C(X, M)$ be the space of continuous mappings from X into M . Also let

$$(8.2) \quad C_b(X, M) = B(X, M) \cap C(X, M)$$

be the space of bounded continuous mappings from X into M . It is easy to see that $C_b(X, M)$ is a closed set in $B(X, M)$ with respect to the supremum q -metric, by standard arguments. If M is complete with respect to $d(\cdot, \cdot)$, then it follows that $C_b(X, M)$ is complete with respect to the supremum q -metric. Note that compact subsets of M are bounded, and hence that continuous mappings from X into M are bounded when X is compact.

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Also let V be a vector space over k , and let N be a q -norm on V with respect to $|\cdot|$ on k . Thus (7.7) defines a q -metric on V , as in the previous section. If X is a nonempty set again, then we shall also use the notation $\ell^\infty(X, V)$ for the space of bounded V -valued functions on X . It is easy to see that this is a vector space over k with respect to pointwise addition and scalar multiplication. Put

$$(8.3) \quad \|f\|_\infty = \|f\|_{\ell^\infty(X, V)} = \sup_{x \in X} N(f(x))$$

for each $f \in \ell^\infty(X, V)$, which defines a q -norm on $\ell^\infty(X, V)$ with respect to $|\cdot|$ on k . This is the *supremum q -norm* on $\ell^\infty(X, V)$ corresponding to N on V . By construction, the q -metric on $\ell^\infty(X, V)$ associated to (8.3) is the supremum q -metric that corresponds to the q -metric (7.7) on V associated to N . If $|\cdot|$ is an ultrametric absolute value function on k , and if N is an ultranorm on V , then (8.3) is an ultranorm on $\ell^\infty(X, V)$ as well. If X is a topological space,

then $C(X, V)$ is a vector space over k with respect to pointwise addition and scalar multiplication too, and $C_b(X, V)$ is a linear subspace of $\ell^\infty(X, V)$. Of course, if X is equipped with the discrete topology, then every function on X is continuous, so that $C_b(X, V)$ is the same as $\ell^\infty(X, V)$.

9 Summable functions

Let X be a nonempty set, and let f be a nonnegative real-valued function on X . The sum

$$(9.1) \quad \sum_{x \in X} f(x)$$

is defined as a nonnegative extended real number to be the supremum of the sums

$$(9.2) \quad \sum_{x \in A} f(x)$$

over all nonempty finite subsets A of X . If g is another nonnegative real-valued function on X and a is a positive real number, then one can check that

$$(9.3) \quad \sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x)$$

and

$$(9.4) \quad \sum_{x \in X} a f(x) = a \sum_{x \in X} f(x),$$

with the usual interpretations for nonnegative extended real numbers. If (9.1) is finite, then f is said to be *summable* on X . If f and g are summable on X , then it follows that $f + g$ is summable on X , and that $a f$ is summable on X for every $a \geq 0$.

Similarly, f is said to be *r -summable* on X for some positive real number r if $f(x)^r$ is summable on X . Put

$$(9.5) \quad \|f\|_r = \left(\sum_{x \in X} f(x)^r \right)^{1/r}$$

when $0 < r < \infty$, and

$$(9.6) \quad \|f\|_\infty = \sup_{x \in X} f(x).$$

Thus (9.5) is finite exactly when f is r -summable on X , and (9.6) is finite exactly when f is bounded on X . If f is bounded on X , then (9.6) is the same as the supremum norm of f , with respect to the standard absolute value function on \mathbf{R} . Note that

$$(9.7) \quad \|a f\|_r = a \|f\|_r$$

for every $a, r > 0$, and in particular that $a f$ is r -summable on X for every $a \geq 0$ when f is r -summable on X .

If f is r -summable on X for some $r > 0$, then it is easy to see that f is bounded on X , and that

$$(9.8) \quad \|f\|_\infty \leq \|f\|_r.$$

This implies that for each $t > r$ and $x \in X$, we have that

$$(9.9) \quad f(x)^t \leq \|f\|_\infty^{t-r} f(x)^r \leq \|f\|_r^{t-r} f(x)^r.$$

Summing over $x \in X$, we get that

$$(9.10) \quad \|f\|_t^t = \sum_{x \in X} f(x)^t \leq \|f\|_r^{t-r} \sum_{x \in X} f(x)^r = \|f\|_r^{t-r} \|f\|_r^r = \|f\|_r^t,$$

and hence

$$(9.11) \quad \|f\|_t \leq \|f\|_r.$$

In particular, f is t -summable on X for every $t > r$ when f is r -summable on X .

Let g be another nonnegative real-valued function on X again, and observe that

$$(9.12) \quad (f(x) + g(x))^r \leq (2 \max(f(x), g(x)))^r \leq 2^r (f(x)^r + g(x)^r)$$

for every $x \in X$ and $r > 0$. If f and g are both r -summable on X , then it follows that $f + g$ is r -summable on X too, by summing over $x \in X$. More precisely, if $0 < r \leq 1$, then we have that

$$(9.13) \quad (f(x) + g(x))^r \leq f(x)^r + g(x)^r$$

for every $x \in X$. This follows from (1.11), with $q_1 = r$ and $q_2 = 1$, and it can also be derived from (9.11), with $t = 1$. Summing both sides of (9.13) over $x \in X$, we get that

$$(9.14) \quad \|f + g\|_r^r \leq \|f\|_r^r + \|g\|_r^r$$

when $0 < r \leq 1$. If $r \geq 1$, then we have that

$$(9.15) \quad \|f + g\|_r \leq \|f\|_r + \|g\|_r,$$

by Minkowski's inequality for sums. Of course, (9.14) and (9.15) reduce to (9.3) when $r = 1$, and it is easy to verify (9.15) directly when $r = \infty$.

10 ℓ^r Norms

Let k be a field, and let $|\cdot|$ be a q -absolute value function on k for some $q > 0$. Also let V be a vector space over k , and let N be a q -norm on V with respect to $|\cdot|$ on k . A V -valued function f on X is said to be r -summable on a nonempty set X for some positive real number r if $N(f(x))$ is r -summable as a nonnegative real-valued function on X , as in the previous section. If f is r -summable with $r = 1$, then we may simply say that f is summable on X . The space of V -valued r -summable functions on X is denoted $\ell^r(X, V)$.

Let f and g be V -valued functions on X , and observe that

$$(10.1) \quad N(f(x) + g(x))^r \leq (N(f(x))^q + N(g(x))^q)^{r/q}$$

for every $x \in X$, by the q -norm version of the triangle inequality. Thus

$$(10.2) \quad N(f(x) + g(x))^r \leq 2^{r/q} (N(f(x))^r + N(g(x))^r)$$

for every $x \in X$, as in (9.12), but with r replaced by r/q . If f and g are both r -summable on X , then it follows that $f + g$ is r -summable too, by summing over $x \in X$. It is easy to see that r -summability is also preserved by scalar multiplication, so that $\ell^r(X, V)$ is a vector space with respect to pointwise addition and scalar multiplication.

Put

$$(10.3) \quad \|f\|_r = \|f\|_{\ell^r(X, V)} = \left(\sum_{x \in X} N(f(x))^r \right)^{1/r}$$

for each $f \in \ell^r(X, V)$, which clearly satisfies the usual positivity and homogeneity requirements of a norm. If $r \leq q$, then (10.1) implies that

$$(10.4) \quad N(f(x) + g(x))^r \leq N(f(x))^r + N(g(x))^r$$

for every $f, g \in \ell^r(X, V)$ and $x \in X$, as in (9.13), with r replaced by r/q . Summing over $x \in X$, we get that

$$(10.5) \quad \|f + g\|_r^r \leq \|f\|_r^r + \|g\|_r^r$$

for every $f, g \in \ell^r(X, V)$, so that $\|f\|_r$ defines an r -norm on $\ell^r(X, V)$ when $r \leq q$. If $q \leq r$, then (10.1) implies that

$$(10.6) \quad \|f + g\|_r^q \leq \|f\|_r^q + \|g\|_r^q$$

for every $f, g \in \ell^r(X, V)$, using (9.15) with r replaced by r/q . This shows that $\|f\|_r$ is a q -norm on $\ell^r(X, V)$ when $q \leq r$.

If N is an ultranorm on V , then we have that

$$(10.7) \quad N(f(x) + g(x))^r \leq \max(N(f(x)), N(g(x)))^r \leq N(f(x))^r + N(g(x))^r$$

for all V -valued functions f, g on X , $r > 0$, and $x \in X$. This implies that (10.5) holds for every $f, g \in \ell^r(X, V)$ and $r > 0$, by summing over $x \in X$. Thus $\|f\|_r$ is an r -norm on $\ell^r(X, V)$ for every $r > 0$ in this case, which corresponds to $q = \infty$ in the previous discussion.

11 Infinite series

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$. Also let V be a vector space over k again, and let N be a q -norm on V with respect to $|\cdot|$ on k . This leads to a q -metric $d(v, w)$ on V associated to N as in (7.7), and

hence to a topology on V , as in Section 2. As usual, an infinite series $\sum_{j=1}^{\infty} a_j$ with terms in V is said to converge in V if the corresponding sequence of partial sums

$$(11.1) \quad s_n = \sum_{j=1}^n a_j$$

converges to an element of V with respect to this topology, in which case the value of the sum $\sum_{j=1}^{\infty} a_j$ is defined to be the limit of the sequence $\{s_n\}_{n=1}^{\infty}$.

As in Section 8, one can define Cauchy sequences and completeness with respect to $d(v, w)$ in the same way as for ordinary metrics, and this is equivalent to defining Cauchy sequences with respect to the metric $d(v, w)^q$. It is easy to see that the sequence (11.1) of partial sums is Cauchy sequence in V for each $\epsilon > 0$ there is a positive integer L such that

$$(11.2) \quad N\left(\sum_{j=l}^n a_j\right) < \epsilon$$

for every $n \geq l \geq L$. In particular, this implies that $\{a_j\}_{j=1}^{\infty}$ converges to 0 in V , by taking $l = n$. Note that

$$(11.3) \quad N\left(\sum_{j=l}^n a_j\right)^q \leq \sum_{j=l}^n N(a_j)^q$$

for every $n \geq j \geq 1$, by the q -norm version of the triangle inequality. Similarly, if N is an ultranorm on V , then

$$(11.4) \quad N\left(\sum_{j=l}^n a_j\right) \leq \max_{l \leq j \leq n} N(a_j)$$

for every $n \geq l \geq 1$.

Let us say that $\sum_{j=1}^{\infty} a_j$ converges *q-absolutely* if

$$(11.5) \quad \sum_{j=1}^{\infty} N(a_j)^q$$

converges as an infinite series of nonnegative real numbers. Of course, this reduces to the usual notion of absolute convergence when $q = 1$. If (11.5) converges, then one can use (11.3) to check that $\sum_{j=1}^{\infty} a_j$ satisfies the Cauchy criterion described in the preceding paragraph, as in the $q = 1$ case. If V is complete, then $\sum_{j=1}^{\infty} a_j$ converges in V , and we have that

$$(11.6) \quad N\left(\sum_{j=1}^{\infty} a_j\right)^q \leq \sum_{j=1}^{\infty} N(a_j)^q,$$

by standard arguments. Similarly, if N is an ultranorm on V , and if $\{a_j\}_{j=1}^{\infty}$ converges to 0 in V , then (11.4) implies that $\sum_{j=1}^{\infty} a_j$ satisfies the Cauchy criterion. If V is complete, then it follows that $\sum_{j=1}^{\infty} a_j$ converges in V , and that

$$(11.7) \quad N\left(\sum_{j=1}^{\infty} a_j\right) \leq \max_{j \geq 1} N(a_j).$$

Note that the maximum on the right side of (11.7) is attained under these conditions, because $N(a_j) \rightarrow 0$ as $j \rightarrow \infty$.

If every q -absolutely convergent series in V converges to an element of V , then a well-known argument implies that V has to be complete. To see this, let $\{v_j\}_{j=1}^{\infty}$ be any Cauchy sequence of elements of V . It is easy to see that there is a subsequence $\{v_{j_l}\}_{l=1}^{\infty}$ of $\{v_j\}_{j=1}^{\infty}$ such that

$$(11.8) \quad N(v_{j_l} - v_{j_{l+1}}) < 2^{-l}$$

for each $l \geq 1$. This implies that $\sum_{l=1}^{\infty} (v_{j_l} - v_{j_{l+1}})$ converges q -absolutely, and hence that this series converges in V , by hypothesis. Of course,

$$(11.9) \quad \sum_{l=1}^n (v_{j_l} - v_{j_{l+1}}) = v_{j_1} - v_{j_{n+1}}$$

for each positive integer n , so that $\sum_{l=1}^{\infty} (v_{j_l} - v_{j_{l+1}})$ converges in V if and only if $\{v_{j_n}\}_{n=1}^{\infty}$ converges as a sequence in V . Because $\{v_j\}_{j=1}^{\infty}$ is a Cauchy sequence in V , the convergence of a subsequence $\{v_{j_n}\}_{n=1}^{\infty}$ in V implies that $\{v_j\}_{j=1}^{\infty}$ converges to the same limit, as desired. If N is an ultranorm on V , then one can consider infinite series $\sum_{j=1}^{\infty} a_j$ with terms in V such that $N(a_j) \rightarrow 0$ as $j \rightarrow \infty$, as the analogue of q -absolute convergence with $q = \infty$. If every such series converges in V , then V has to be complete, as before. In this case, if $\{v_j\}_{j=1}^{\infty}$ is any Cauchy sequence in V , then one can apply the hypothesis on infinite series directly to $\sum_{j=1}^{\infty} (v_j - v_{j+1})$.

12 Vanishing at infinity

Let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$ again, let V be a vector space over k , and let N be a q -norm on V with respect to $|\cdot|$. Also let X be a (nonempty) locally compact Hausdorff topological space. A continuous V -valued function f on X is said to *vanish at infinity* if for each $\epsilon > 0$ there is a compact set $K(\epsilon) \subseteq X$ such that

$$(12.1) \quad N(f(x)) < \epsilon$$

for every $x \in X \setminus K(\epsilon)$. This is equivalent to saying that

$$(12.2) \quad \{x \in X : N(f(x)) \geq \epsilon\}$$

is a compact subset of X for each $\epsilon > 0$. More precisely, if (12.2) is compact for some $\epsilon > 0$, then one can simply take $K(\epsilon)$ to be (12.2). Conversely, if $K(\epsilon)$ is a compact subset of X such that (12.1) holds for every $x \in X \setminus K(\epsilon)$, then (12.2) is contained in $K(\epsilon)$. If f is continuous, then (12.2) is a closed set in X , since it is the inverse image of a closed ball in V . This implies that (12.2) is compact, because closed subsets of compact sets are compact.

The space of continuous V -valued functions on X that vanish at infinity is denoted $C_0(X, V)$. It is easy to see that

$$(12.3) \quad C_0(X, V) \subseteq C_b(X, V),$$

by taking $\epsilon = 1$ in the previous definition, and using the fact that continuous functions are bounded on compact sets. Moreover, $C_0(X, V)$ is a linear subspace of $C_b(X, V)$, as a vector space with respect to pointwise addition and scalar multiplication. One can also check that $C_0(X, V)$ is a closed set in $C_b(X, V)$, with respect to the topology determined by the supremum q -norm.

If f is any V -valued function on X , then the *support* is denoted $\text{supp } f$, and defined to be the closure in X of the set of $x \in X$ such that $f(x) \neq 0$. The space of continuous V -valued functions with compact support in X may be denoted $C_{\text{com}}(X, V)$ or $C_{00}(X, V)$, and is a linear subspace of $C_0(X, V)$. If X is equipped with the discrete topology, so that every function on X is continuous, then $C_0(X, V)$ may also be denoted $c_0(X, V)$, and $C_{00}(X, V)$ may be denoted $c_{00}(X, V)$. In this case, the support of a V -valued function f on X is simply the set of $x \in X$ such that $f(x) \neq 0$, and the only compact subsets of X are those with only finitely many elements. Thus $c_{00}(X, V)$ consists of the V -valued functions f on X such that $f(x) = 0$ for all but finitely many $x \in X$, and $c_0(X, V)$ consists of the V -valued functions f on V such that for each $\epsilon > 0$, (12.1) holds for all but finitely many $x \in X$.

Let X be a locally compact Hausdorff topological space again. If $K \subseteq X$ is compact, $U \subseteq X$ is an open set, and $K \subseteq U$, then it is well known that there is a continuous real-valued function on X with compact support contained in U which is equal to 1 on K , and which takes values between 0 and 1 on all of X , by Urysohn's lemma. If $k = \mathbf{R}$ or \mathbf{C} equipped with the standard absolute value function, then one can use this to show that $C_{\text{com}}(X, V)$ is dense in $C_0(X, V)$ with respect to the supremum q -norm. Of course, the same argument can be used when $k = \mathbf{R}$ or \mathbf{C} is equipped with a q -absolute value function which is a power of the standard absolute value function.

If K_1 is a compact open subset of X , then the function on X equal to 1 on K_1 and to 0 on $X \setminus K_1$ is continuous and has compact support equal to K_1 . If every compact subset of X is contained in a compact open set, then one can use these functions to show that $C_{\text{com}}(X, V)$ is dense in $C_0(X, V)$ with respect to the supremum q -norm. More precisely, this works for any field k with a q -absolute value function, and for any vector space V over k with a q -norm N . In particular, this condition holds when X is equipped with the discrete topology. Note that this condition also holds when X is locally compact and has topological dimension 0, as in Section 14.

Suppose for the moment that $|\cdot|$ is an ultrametric absolute value function on k , and that N is an ultranorm on V . Thus

$$(12.4) \quad \{v \in V : N(v) \geq \epsilon\}$$

is an open set in V with respect to the topology determined by the ultrametric associated to N for every $\epsilon > 0$, as in Section 2. If $f : X \rightarrow V$ is continuous, then it follows that (12.2) is an open set in X for every $\epsilon > 0$, since (12.2) is the same as the inverse image of (12.4) under f . If f also vanishes at infinity on X , then we have seen that (12.2) is a compact subset of X for every $\epsilon > 0$ too. Using this and the remarks at the beginning of the previous paragraph, one can check that $C_{com}(X, V)$ is dense in $C_0(X, V)$ with respect to the supremum norm in this case as well.

Let k be any field with a q -absolute value function $|\cdot|$ again, and let V be a vector space with a q -norm N . Also let X be a nonempty set, which may be considered as being equipped with the discrete topology, and let r be a positive real number. Observe that every V -valued function on X with finite support is r -summable, so that

$$(12.5) \quad c_{00}(X, V) \subseteq \ell^r(X, V).$$

If $f \in \ell^r(X, V)$, then

$$(12.6) \quad \sum_{x \in X} N(f(x))^r < \infty,$$

where the sum is defined as the supremum of the corresponding finite subsums, as in Section 9. Thus for each $\epsilon > 0$ there should be a finite set $A(\epsilon) \subseteq X$ such that

$$(12.7) \quad \sum_{x \in X} N(f(x))^r < \sum_{x \in A(\epsilon)} N(f(x))^r + \epsilon,$$

which implies that

$$(12.8) \quad \sum_{x \in X \setminus A(\epsilon)} N(f(x))^r < \epsilon.$$

It follows from this that f can be approximated by V -valued functions with finite support in X with respect to the ℓ^r norm, so that $c_{00}(X, V)$ is dense in $\ell^r(X, V)$. In particular, this argument shows that f vanishes at infinity on X , which implies that

$$(12.9) \quad \ell^r(X, V) \subseteq c_0(X, V).$$

Part II

Topological dimension

13 Separation conditions

Remember that a topological space X satisfies the *first separation condition* if for each $x, y \in X$ with $x \neq y$ there is an open subset of X that contains x and

not y . This implies that there is also an open subset of X that contains y and not x , by interchanging the roles of x and y . Equivalently, X satisfies the first separation condition if and only if every subset of X with exactly one element is a closed set, which implies that finite subsets of X are closed sets. Similarly, X satisfies the *second separation condition* if every pair of distinct elements of X is contained in a pair of disjoint open subsets of X . This obviously implies that X satisfies the first separation condition, and topological spaces that satisfy the second separation condition are said to be *Hausdorff*.

The *0th separation condition* asks that for each pair of distinct elements of X there be an open subset of X that contains one of the two points and not the other, but without specifying which of the two points is contained in the open set. Thus the first separation condition automatically implies the 0th separation condition. Equivalently, X satisfies the 0th separation condition if for every pair of distinct elements of X there is a closed set in X that contains one of the two points and not the other, without specifying which of the two points is contained in the closed set.

If X satisfies the 0th separation condition, and for each $x \in X$ and closed set $E \subseteq X$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $E \subseteq V$, then X satisfies the *third separation condition*, and is said to be *regular*. It is easy to see that regular topological spaces are Hausdorff, and in particular that they satisfy the first separation condition. More precisely, if $x, y \in X$ and $x \neq y$, then the 0th separation condition implies that there is a closed set in X that contains one of x, y and not the other, and one can use the rest of the regularity condition to show that x, y are contained in disjoint open subsets of X . Sometimes regularity of topological spaces is defined by including the first separation condition in the definition instead of the 0th separation condition, which would be equivalent by the previous remarks. Regularity can also be characterized by asking that X satisfy the 0th separation condition, and that for every $x \in X$ and open set $W \subseteq X$ with $x \in W$ there be an open set $U \subseteq X$ such that $x \in U$ and $\overline{U} \subseteq W$, where \overline{U} denotes the closure of U in X .

A topological space X is said to be *completely Hausdorff* if every pair of distinct elements of X is contained in a pair of open subsets of X with disjoint closures in X . This is also known as separation condition number two and a half. Completely Hausdorff spaces are obviously Hausdorff, and regular topological spaces are completely Hausdorff.

If X satisfies the first separation condition, and if every pair of disjoint closed subsets of X are contained in disjoint open subsets of X , then X satisfies the *fourth separation condition*. This implies that X satisfies the second and third separation conditions, and X is said to be *normal* in this case. Equivalently, X is normal if X satisfies the first separation condition, and for every closed set $A \subseteq X$ and open set $W \subseteq X$ with $A \subseteq W$ there is an open set $U \subseteq X$ such that $A \subseteq U$ and $\overline{U} \subseteq W$.

Remember that a pair of subsets A, B of a topological space X are said to be *separated* in X if

$$(13.1) \quad \overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

If X satisfies the first separation condition, and if every pair of separated subsets of X are contained in disjoint open subsets of X , then X satisfies the *fifth separation condition*, and X is said to be *completely normal*. Completely normal topological spaces are automatically normal, because disjoint closed sets are obviously separated. It is well known that metric spaces are completely normal.

Let Y be a subset of a topological space X , equipped with the induced topology. If X satisfies any of the 0th, first, second, or third separation conditions, then Y has the same property. This also works for completely Hausdorff and completely normal spaces, but not for normal spaces. In the case of completely normal spaces, this uses the fact that a pair of subsets of Y are separated with respect to the induced topology on Y if and only if they are separated as subsets of X .

Let τ_1 and τ_2 be topologies on a set X with $\tau_1 \subseteq \tau_2$. If X satisfies any of the 0th, first, or second separation conditions with respect to τ_1 , then X has the same property with respect to τ_2 . This also works for the completely Hausdorff condition, but not for regularity.

If X is a Hausdorff topological space and $K \subseteq X$ is compact, then it is well known that K is a closed set in X . More precisely, if $x \in X \setminus K$, then x and K are contained in disjoint open subsets of X . To see this, one can use the Hausdorff condition to cover K by open sets, each of which is disjoint from an open set that contains x , and then use compactness to reduce to a finite subcovering. Similarly, one can show that every pair of disjoint compact subsets of X is contained in a pair of disjoint open sets. In particular, this implies that compact Hausdorff spaces are normal, because closed subsets of compact spaces are compact as well. If X is regular, $E \subseteq X$ is a closed set, $K \subseteq X$ is compact, and $E \cap K = \emptyset$, then E and K are contained in disjoint open subsets of X . This can be obtained by covering K by open sets, each of which is disjoint from an open set that contains E , and using compactness to reduce to a finite subcovering.

A topological space X is said to be *locally compact* if for each $x \in X$ there is an open set $W \subseteq X$ and a compact set $K \subseteq X$ such that $x \in W$ and $W \subseteq K$. If X is also Hausdorff, then K is a closed set in X , so that $\overline{W} \subseteq K$. This implies that \overline{W} is compact, since closed subsets of compact sets are compact. If X is locally compact and $H \subseteq X$ is compact, then it is easy to see that H is contained in an open set in X that is contained in another compact set, by covering H by finitely many open sets that are contained in compact sets.

Suppose that X is a Hausdorff topological space, $W \subseteq X$ is an open set, $x \in W$, and \overline{W} is compact. Thus the boundary $\partial W = \overline{W} \setminus W$ of W is compact, and of course $x \notin \partial W$. As before, there is an open set $U \subseteq X$ that contains x and is disjoint from an open set that contains ∂W , which means that

$$(13.2) \quad \overline{U} \cap \partial W = \emptyset.$$

If $U_1 = U \cap W$, then U_1 is an open set in X that contains x and satisfies $\overline{U_1} \subseteq \overline{U} \cap \overline{W}$, which implies that

$$(13.3) \quad \overline{U_1} \subseteq W,$$

by (13.2). Using this, one can check that locally compact Hausdorff spaces are regular, since one can always replace W with a smaller open set if necessary to get that \overline{W} is compact.

14 Dimension 0

A subset E of a topological space X is said to be *connected* in X if E cannot be expressed as the union of two nonempty separated sets in X . If $E \subseteq Y \subseteq X$, then E is connected in X if and only if E is connected in Y , with respect to the induced topology on Y . This follows from the analogous statement for separated sets, which was mentioned in the previous section. As before, disjoint closed sets in X are automatically separated, and disjoint open subsets of X are separated too. If $A, B \subseteq X$ are separated and $A \cup B = X$, then A and B are both open and both closed. Thus X is connected if and only if it cannot be expressed as the union of two disjoint nonempty open sets, which is equivalent to saying that X cannot be expressed as the union of two disjoint nonempty closed sets. A set $E \subseteq X$ is said to be *totally disconnected* if it does not contain any connected sets with at least two elements.

A topological space X is said to be *totally separated* if for every $x, y \in X$ with $x \neq y$ there are disjoint open subsets U, V of X such that $x \in U, y \in V$, and $U \cup V = X$. Note that U and V are also closed sets in X under these conditions, so that totally separated spaces are completely Hausdorff. If X is totally separated and $Y \subseteq X$, then Y is totally separated with respect to the induced topology. If τ_1 and τ_2 are topologies on a set X such that $\tau_1 \subseteq \tau_2$, and if X is totally separated with respect to τ_1 , then X is totally separated with respect to τ_2 as well. Totally separated spaces are totally disconnected, which can be derived from the previous statement about subspaces of totally separated spaces and the fact that totally separated spaces with at least two elements are not connected.

A topological space X is said to have *topological dimension 0* at a point $x \in X$ if for every open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq W$ such that $x \in U, U \subseteq W$, and U is also a closed set in X . Of course, this is the same as saying that there is a local base for the topology of X at x consisting of subsets of X that are both open and closed. Similarly, X is said to have topological dimension 0 if X has topological dimension 0 at every point $x \in X$, which is the same as saying that there is a base for the topology of X consisting of sets that are both open and closed. One may also ask that X be nonempty, and define the topological dimension of the empty set to be -1 . Ultrametric spaces have topological dimension 0, because open and closed balls of positive radius are both open and closed with respect to the corresponding topology, as in Section 2.

If X has topological dimension 0 and $Y \subseteq X$, then one can check that Y has topological dimension 0 with respect to the induced topology. More precisely, if the topological dimension of the empty set is defined to be -1 , then one should ask that $Y \neq \emptyset$ too. If X satisfies the 0th separation condition

and has topological dimension 0, then it is easy to see that X is regular as a topological space. In this case, X is totally separated, and in particular X is totally disconnected.

Suppose that X is totally separated, $K \subseteq X$ is compact, and $x \in X \setminus K$. If $y \in K$, then $y \neq x$, and so there is an open set $V(y) \subseteq X$ that is also closed such that $y \in V(y)$ and $x \notin V(y)$, because X is totally separated. It follows that K can be covered by finitely many of these sets $V(y)$, because K is compact, which leads to an open set $V \subseteq X$ such that $K \subseteq V$, $x \notin V$, and V is a closed set too. Equivalently, $U = X \setminus V$ is an open set that is closed as well and satisfies $x \in U$ and $K \cap U = \emptyset$. If H and K are disjoint compact subsets of X , then one can repeat the process to get a subset of X that is both open and closed, and which contains H and is disjoint from K .

If $X \neq \emptyset$ is totally separated and compact, then X has topological dimension 0. To see this, let $x \in X$ and an open set $W \subseteq X$ be given, with $x \in W$. Thus $X \setminus W$ is a closed set in X , so that $X \setminus W$ is compact, because X is compact. As in the previous paragraph, there is a set $U \subseteq X$ that is both open and closed, which contains x , and is disjoint from $X \setminus W$. Hence $U \subseteq W$, as desired.

As another variant of this type of argument, suppose that X has topological dimension 0, $K \subseteq X$ is compact, and that $W \subseteq X$ is an open set that contains K . Thus each element of K is contained in a subset of W that is both open and closed in X . It follows that K is contained in a subset of W that is both open and closed in X , using compactness of K to reduce to a finite subcovering.

If X is locally compact and has topological dimension 0, then for each $x \in X$ and open set $W \subseteq X$ with $x \in W$ there is an open set $U \subseteq X$ such that $x \in U$, $U \subseteq W$, and U is also closed and compact. More precisely, if X is locally compact, then we can always replace W by a smaller open set that contains x and is contained in a compact set. This implies that U is compact in this situation, since it is a closed set contained in a compact set. Similarly, if X is locally compact and has topological dimension 0, $H \subseteq X$ is compact, and $W \subseteq X$ is an open set such that $H \subseteq W$, then H is contained in a subset of W that is open, closed, and compact.

Suppose that X is totally separated, $W \subseteq X$ is an open set, $x \in W$, and \overline{W} is compact. This implies that ∂W is a compact set that does not contain x , so that there is an open set $U \subseteq X$ that is also closed, contains x , and satisfies

$$(14.1) \quad U \cap \partial W = \emptyset,$$

as before. Thus $U_1 = U \cap W$ is an open set in X that contains x and satisfies $\overline{U_1} \subseteq U \cap \overline{W}$, which implies that

$$(14.2) \quad \overline{U_1} \subseteq U \cap W = U_1,$$

by (14.1). Of course, this means that U_1 is a closed set too. It follows that a nonempty totally separated locally compact topological space has topological dimension 0, since one can replace W with a smaller open set to get \overline{W} to be compact, as usual.

15 Chain connectedness

Suppose that M is a nonempty set equipped with a q -metric $d(x, y)$ for some $q > 0$, which leads to a topology on M , as in Section 2. Of course, one can always reduce to the case of ordinary metrics, using $d(x, y)^q$ when $q < 1$. Let us say that $A, B \subseteq M$ are η -separated in M for some $\eta > 0$ if

$$(15.1) \quad d(x, y) \geq \eta$$

for every $x \in A$ and $y \in B$. This implies that A and B are separated in the usual topological sense, and in fact that the closures of A and B are disjoint. In the other direction, if $A, B \subseteq M$ are separated in the topological sense, and if at least one of A and B is compact, then A and B are η -separated for some $\eta > 0$. More precisely, if A is compact, then one may as well suppose that B is a closed set, since otherwise one can replace B with its closure. The initial statement can be shown using standard arguments without this observation, but it is perhaps more commonly given in this way.

A finite sequence w_1, \dots, w_n of elements of M is said to be an η -chain for some $\eta > 0$ if

$$(15.2) \quad d(w_j, w_{j+1}) < \eta$$

for each j with $1 \leq j < n$, which is vacuous when $n = 1$. Put

$$(15.3) \quad x \sim_\eta y$$

when $x, y \in M$ can be connected by an η -chain in M , which is to say that there is an η -chain w_1, \dots, w_n of elements of M with $x = w_1$ and $y = w_n$. It is easy to see that this defines an equivalence relation on M , which leads to a partition of M into equivalence classes. Each of these equivalence classes is an open set in M , and in fact each equivalence class associated to (15.3) contains the open ball of radius η in M centered at any element of the equivalence class. Any two distinct equivalence classes associated to (15.3) are η -separated in M .

Let us say that M is η -connected if every pair of elements of M can be connected by an η -chain of elements of M . If M is not η -connected, then M can be expressed as the union of two nonempty η -separated subsets of M . More precisely, if M is not η -connected, then there are points $x, y \in M$ that cannot be connected by an η -chain of elements of M . Let A be the set of points in M that can be connected to x by an η -chain of elements of M , and put $B = M \setminus A$. Thus $x \in A$, $y \in B$, $A \cup B = M$, and one can check that A and B are η -separated in M . Conversely, if A, B are η -separated subsets of M such that $A \cup B = M$, then there is no η -chain of elements of M that connects a point in A to a point in B . This is because such an η -chain would have to go directly from an element of A to an element of B at some step, which is not possible if A and B are η -separated in M . It follows that M is not η -connected when M can be expressed as the union of two nonempty η -separated sets.

Similarly, a set $E \subseteq M$ is said to be η -connected if every pair of elements of E can be connected by an η -chain of elements of E . Equivalently, E is η -connected if E cannot be expressed as the union of two nonempty η -separated

sets. This follows from the discussion in the previous paragraph when $E = M$. Otherwise, one can reduce to that case, because E is η -connected as a subset of M if and only if E is η -connected as a subset of itself, using the restriction of $d(x, y)$ to $x, y \in E$.

If E is η -connected for every $\eta > 0$, then we say that $E \subseteq M$ is *chain connected*. Thus if E is not chain connected, then E is not η -connected for some $\eta > 0$, so that E can be expressed as the union of two nonempty η -separated sets. This implies that E is not connected, since η -separated sets are separated in the usual sense. It follows that connected subsets of M are chain connected. In the other direction, if $E \subseteq M$ is compact and not connected, then E can be expressed as the union of two nonempty separated sets A and B , and one can check that A and B also have to be compact in this case. This implies that A and B are η -separated for some $\eta > 0$, as mentioned earlier, so that E is not η -connected. Hence compact chain-connected subsets of M are connected.

Let us say that M is *strongly totally separated* if for each $x, y \in M$ with $x \neq y$ there are an $\eta > 0$ and η -separated sets $U, V \subseteq M$ such that $x \in U$, $y \in V$, and $U \cup V = M$. Note that U and V have to be open subsets of M under these conditions, since they are separated and their union is equal to M . Thus M is totally separated when M is strongly totally separated. Equivalently, M is strongly totally separated if for each $x, y \in M$ with $x \neq y$ there is an $\eta > 0$ such that x and y cannot be connected by an η -chain of elements of M , as in the earlier discussion of η -connectedness. If M is strongly totally separated and $Y \subseteq M$, then it is easy to see that Y is strongly totally separated too, with respect to the restriction of $d(x, y)$ to $x, y \in Y$.

Similarly, let us say that M is *strongly 0-dimensional* if for each $x \in M$ and $r > 0$ there is an open set $U \subseteq M$ such that $x \in U$, $U \subseteq B(x, r)$, and U , $M \setminus U$ are η -separated for some $\eta > 0$. This implies that M is strongly totally separated, and that M has topological dimension 0. As before, one may wish to require that M be nonempty in order to be strongly 0-dimensional, in particular to be consistent in the second part of the preceding statement. If M is strongly 0-dimensional and $Y \subseteq M$, then Y is strongly 0-dimensional with respect to the restriction of $d(x, y)$ to $x, y \in Y$. If nonemptiness is included in the definition of strongly 0-dimensional spaces, then one should also ask that Y be nonempty in the previous statement.

If M has topological dimension 0 and is locally compact, then M is strongly 0-dimensional. This uses the fact that if $U \subseteq M$ is compact and open, then U and $M \setminus U$ are η -separated for some $\eta > 0$. Note that the set \mathbf{Q} of rational numbers has topological dimension 0 with respect to the standard topology, even though \mathbf{Q} is chain connected with respect to the standard metric on \mathbf{R} . If $d(x, y)$ is an ultrametric on a set M , then M is strongly 0-dimensional with respect to the corresponding topology, because $B(x, r)$ and $M \setminus B(x, r)$ are r -separated for every $x \in M$ and $r > 0$.

Suppose that M is strongly totally separated, $K \subseteq M$ is compact, and x is an element of $M \setminus K$. Using a covering argument as in the previous section, one can check that there is an open set $V \subseteq M$ such that $K \subseteq V$, $x \notin V$, and V , $M \setminus V$ are η -separated for some $\eta > 0$. If H , K are disjoint compact

subsets of M , then one can repeat the process to get an open set $U \subseteq M$ such that $H \subseteq U$, $U \cap K = \emptyset$, and U , $M \setminus U$ are η -separated for some $\eta > 0$. If M is strongly 0-dimensional, $K \subseteq M$ is compact, $W \subseteq M$ is an open set, and $K \subseteq W$, then an analogous argument implies that there is an open set $U \subseteq M$ such that $K \subseteq U$, $U \subseteq W$, and U , $M \setminus U$ are η -separated for some $\eta > 0$. If M is also locally compact, then one can take U to be compact as well.

16 ℓ^r Spaces

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let V be a vector space over k with an ultranorm N with respect to $|\cdot|$ on k . Also let X be a nonempty set, so that $\ell^r(X, V)$ can be defined as in Sections 8 and 10 for $0 < r \leq \infty$. Under these conditions, $\|f\|_r$ defines an r -norm on $\ell^r(X, V)$, which leads to an r -metric on $\ell^r(X, V)$, as in Section 7. In particular, the supremum norm defines an ultranorm on $\ell^\infty(X, V)$ in this situation, and the corresponding supremum metric is an ultrametric. Thus $\ell^\infty(X, V)$ is strongly 0-dimensional with respect to the supremum metric, as in the previous section.

Suppose from now on in this section that $0 < r < \infty$. Remember that r -summable functions are bounded on X , so that

$$(16.1) \quad \ell^r(X, V) \subseteq \ell^\infty(X, V).$$

This implies that $\ell^r(X, V)$ is strongly 0-dimensional with respect to the supremum metric, because of the analogous property of $\ell^\infty(X, V)$. Similarly,

$$(16.2) \quad \|f\|_\infty \leq \|f\|_r$$

for every $f \in \ell^r(X, V)$, which means that the r -metric on $\ell^r(X, V)$ associated to $\|f\|_r$ is greater than or equal to the supremum metric. If $A, B \subseteq \ell^r(X, V)$ are η -separated with respect to the supremum metric for some $\eta > 0$, then it follows that A and B are also η -separated with respect to the r -metric associated to $\|f\|_r$. Of course, $\ell^\infty(X, V)$ is strongly totally separated with respect to the supremum metric, since it is strongly 0-dimensional. It follows that $\ell^r(X, V)$ is also strongly totally separated with respect to the supremum metric, and hence with respect to the r -metric associated to $\|f\|_r$.

Suppose for the moment that $|\cdot|$ is the trivial absolute value function on k , and that N is the trivial ultranorm on V . This implies that the supremum norm is the trivial ultranorm on $\ell^\infty(X, V)$, so that the supremum metric on $\ell^\infty(X, V)$ is the same as the discrete metric. Note that every r -summable V -valued function on X has finite support in X in this case, and hence

$$(16.3) \quad \ell^r(X, V) = c_{00}(X, V).$$

Using (16.2), we get that the r -metric associated to $\|f\|_r$ on $\ell^r(X, V)$ is greater than or equal to the discrete metric, which implies that the topology on $\ell^r(X, V)$ determined by the r -metric associated to $\|f\|_r$ is the discrete topology. It is

easy to see that $\ell^r(X, V)$ is strongly 0-dimensional with respect to the r -metric associated to $\|f\|_r$ in this situation.

Let $|\cdot|$ be any ultrametric absolute value function on a field k again, and let N be any ultranorm on a vector space V over k . If X is a finite set with n elements, then every V -valued function f on X is r -summable, and satisfies

$$(16.4) \quad \|f\|_r \leq n^{1/r} \|f\|_\infty.$$

Of course, this leads to a similar relationship between the r -metric on $\ell^r(X, V) = \ell^\infty(X, V)$ associated to $\|f\|_r$ and the supremum metric. It follows that $\ell^r(X, V)$ is strongly 0-dimensional with respect to the r -metric associated to $\|f\|_r$ in this situation, because of the analogous property of $\ell^\infty(X, V)$ with respect to the supremum metric.

Let us suppose from now on in this section that $|\cdot|$ is nontrivial on k , $V \neq \{0\}$, and that X has infinitely many elements. As in Section 4, the nontriviality of $|\cdot|$ on k means that there are nonzero elements of k with absolute value strictly less than 1. This implies that there are nonzero elements of k with arbitrarily small absolute value, by taking large integer powers of the previous elements. It follows that there are nonzero elements of V with arbitrarily small norm, because $V \neq \{0\}$.

Let $\eta > 0$ be given, and let v_η be a nonzero element of V with

$$(16.5) \quad N(v_\eta) < \eta,$$

as in the preceding paragraph. Also let x_1, \dots, x_n be finitely many distinct elements of X . If j is a positive integer less than or equal to n , then let $a_j(x)$ be the V -valued function on X defined by putting

$$(16.6) \quad a_j(x_j) = v_\eta$$

and $a_j(x) = 0$ when $x \neq x_j$. Put

$$(16.7) \quad f_l(x) = \sum_{j=1}^l a_j(x)$$

for each $l = 1, \dots, n$ and $x \in X$, and $f_0(x) = 0$ for every $x \in X$. Thus $f_l \in c_{00}(X, V) \subseteq \ell^r(X, V)$ for each $l = 0, 1, \dots, n$, and

$$(16.8) \quad \|f_l - f_{l-1}\|_r = \|a_l\|_r = N(v_\eta)$$

when $l \geq 1$. This shows that f_0, f_1, \dots, f_n is an η -chain in $c_{00}(X, V)$ with respect to the r -metric associated to $\|f\|_r$. We also have that

$$(16.9) \quad \|f_l\|_r = l^{1/r} N(v_\eta)$$

for each l , because the x_j 's are supposed to be distinct elements of X .

Suppose that U is a nonempty subset of $\ell^r(X, V)$ which is η -separated from its complement in $\ell^r(X, V)$ with respect to the r -metric associated to $\|f\|_r$.

This means that if an η -chain of elements of $\ell^r(X, V)$ with respect to this r -metric starts at an element of U , then this η -chain should stay in U at every step. Using the η -chains described in the previous paragraph, one can check that this implies that U is unbounded with respect to the r -metric associated to $\|f\|_r$. It is convenient to reduce to the case where $0 \in U$, although this is not really necessary. This also uses the hypothesis that X have infinitely many elements, so that the η -chain can have arbitrary length. This implies that $\ell^r(X, V)$ is not strongly 0-dimensional with respect to the r -metric associated to $\|f\|_r$ under these conditions. Similarly, $c_{00}(X, V)$ is not strongly 0-dimensional with respect to this r -metric.

It is easy to see that $\ell^2(\mathbf{Z}_+, \mathbf{Q})$ is totally separated with respect to the ℓ^2 metric, using the restriction of the standard absolute value function on \mathbf{R} to \mathbf{Q} . A well-known theorem of Erdős implies that $\ell^2(\mathbf{Z}_+, \mathbf{Q})$ does not have topological dimension 0, as in Example II 11 on p13 of [8]. This argument seems to carry over nicely to $\ell^r(X, V)$, under the same conditions as before, with some adjustments. In both situations, it suffices to show that a bounded open set U that contains 0 has nonempty boundary. To do this, one looks for a sequence of elements of U for which the distance to the complement converges to 0, and where the sequence converges in the space being considered. In the classical case of $\ell^2(\mathbf{Z}_+, \mathbf{Q})$, the n th term of the sequence has at most n nonzero coordinates, and one modifies the next coordinate to get closer to the complement of U . Similarly, in the context of $\ell^r(X, V)$, each term in the sequence has only finitely many nonzero coordinates, and each successive term modifies only finitely many coordinates that have not been changed previously. In these finitely many new coordinates, one can use η -chains of the same type as before. More precisely, one adds an η -chain without leaving U , but where a single additional step in the η -chain would leave U . This is possible, because U is bounded, by hypothesis, and this ensures that the resulting element of U is as close to the complement of U as one wants, by taking η to be sufficiently small. As in the classical case, it is easy to see that a sequence of elements of U constructed in this way converges to an element of $\ell^r(X, V)$. This uses the hypothesis that U be bounded, so that the terms in the sequence have bounded norm, and the fact that each new term in the sequence only changes the coordinates that were equal to 0 before, by construction. Of course, the limit of the sequence is an element of the boundary of U , so that the boundary of U is nonempty, as desired.

17 Some variants

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k . Also let X be a nonempty set, let r be a positive real number, and let a be a nonnegative real-valued function on X which is r -summable. If f is a V -valued function on X that satisfies

$$(17.1) \quad N(f(x)) \leq a(x) \quad \text{for every } x \in X,$$

then f is r -summable on X too. Let E_a be the set of $f : X \rightarrow V$ that satisfy (17.1), so that $E_a \subseteq \ell^r(X, V)$. This is the same as the classical Hilbert cube when $V = k = \mathbf{R}$ with the standard absolute value function, $X = \mathbf{Z}_+$, $r = 2$, and $a(j) = 1/j$ for each $j \in \mathbf{Z}_+$. Suppose for the moment that $V = k = \mathbf{Q}$, equipped with the restriction of the standard absolute value function on \mathbf{R} to \mathbf{Q} . Remember that $\|f\|_r$ defines a norm on $\ell^r(X, \mathbf{Q})$ when $r \geq 1$, and an r -norm when $0 < r \leq 1$, as in Section 10. One can check that E_a has topological dimension 0 with respect to the topology determined by the metric or r -metric corresponding to $\|f\|_r$. This corresponds to Example II 9 on p12 of [8] when $X = \mathbf{Z}_+$, $r = 2$, and $a(j) = 1/j$ for each $j \in \mathbf{Z}_+$, and essentially the same argument can be used otherwise.

Suppose now that $|\cdot|$ is an ultrametric absolute value function on a field k , and that N is an ultranorm on a vector space V over k . Thus $\|f\|_r$ defines an r -norm on $\ell^r(X, V)$, as in Section 10. In this case, one can check that E_a is strongly 0-dimensional with respect to the r -metric associated to $\|f\|_r$. This uses the fact that for each $\epsilon > 0$ there is a finite set $A(\epsilon) \subseteq X$ such that

$$(17.2) \quad \sum_{x \in X \setminus A(\epsilon)} a(x)^r < \epsilon,$$

as in (12.8). Of course, every subset of $\ell^r(X, V)$ is strongly totally separated with respect to the r -metric associated to $\|f\|_r$ under these conditions, since $\ell^r(X, V)$ is strongly totally separated with respect to this r -metric, as in the previous section.

Let us continue to suppose that $|\cdot|$ be an ultrametric absolute value function on k , and that N be an ultranorm on V . Also let X be a nonempty set, and let r be a positive real number, as before. Consider the vector space $c_{00}(X, V)$ of V -valued functions on X with finite support, equipped with the r -metric associated to $\|f\|_r$. Suppose for the moment that N takes values on V in a set of finitely or countably many nonnegative real numbers. This implies that $N(v)^r$ also takes values in a set of finitely or countably many nonnegative real numbers, and hence that the collection of all finite sums of elements of this set has only finitely or countably many elements. It follows that $\|f\|_r$ takes only finitely or countably many values on $c_{00}(X, V)$, which implies that the corresponding r -metric only takes finitely or countably many values on $c_{00}(X, V)$ as well. In this case, $c_{00}(X, V)$ has topological dimension 0 with respect to the topology determined by the r -metric associated to $\|f\|_r$, since open and closed balls of all but finitely or countably many radii are automatically the same.

Otherwise, if N does not take values in a set of nonnegative real numbers with only finitely or countably many elements, then we can basically reduce to this case by modifying N . More precisely, let $h(t)$ be a monotonically increasing real-valued function defined on the set of nonnegative real numbers such that $h(0) = 0$ and $h(t) > 0$ when $t > 0$. Under these conditions, it is easy to see that

$$(17.3) \quad h(N(v - w))$$

defines an ultrametric on V which determines the same topology on V as the

ultrametric $N(v - w)$ associated to N . Put

$$(17.4) \quad d_r(f, g) = \left(\sum_{x \in X} h(N(f(x) - g(x)))^r \right)^{1/r}$$

for every $f, g \in c_{00}(X, V)$, which defines an r -metric on $c_{00}(X, V)$, for the same reasons as in Section 10. If $h(t)$ and t are each bounded by positive constant multiples of the other on $[0, +\infty)$, then (17.4) and $\|f - g\|_r$ are each bounded by the same constant multiples of the other on $c_{00}(X, V)$. In particular, this implies that these two r -metrics determine the same topology on $c_{00}(X, V)$. We can also choose h so that it takes values in a countable subset of \mathbf{R} , which implies that (17.4) takes values in a countable set of nonnegative real numbers when $f, g \in c_{00}(X, V)$. As before, this means that open and closed balls in $c_{00}(X, V)$ with respect to (17.4) of all but finitely or countably many radii are the same, and hence that $c_{00}(X, V)$ has topological dimension 0 with respect to the corresponding topology.

18 ℓ^r Spaces, continued

Let k be a field with an absolute value function $|\cdot|$ again, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k . Also let X be a nonempty set, let r and t be positive real numbers, and let f be an r -summable V -valued function on X with

$$(18.1) \quad \|f\|_r = \left(\sum_{x \in X} N(f(x))^r \right)^{1/r} = t.$$

Thus for each $\epsilon > 0$ there is a finite set $A(\epsilon) \subseteq X$ such that

$$(18.2) \quad \sum_{x \in A(\epsilon)} N(f(x))^r > \sum_{x \in X} N(f(x))^r - \epsilon = t^r - \epsilon,$$

as in (12.7). As before, this implies that

$$(18.3) \quad \sum_{x \in X \setminus A(\epsilon)} N(f(x))^r < \epsilon.$$

Suppose that g is another V -valued function on X that is sufficiently close to f on $A(\epsilon)$ so that

$$(18.4) \quad \sum_{x \in A(\epsilon)} N(g(x))^r > t^r - 2\epsilon.$$

If we also have that $g \in \ell^r(X, V)$ satisfies

$$(18.5) \quad \|g\|_r = \left(\sum_{x \in X} N(g(x))^r \right)^{1/r} \leq t,$$

then it follows that

$$\begin{aligned}
(18.6) \quad \sum_{x \in X \setminus A(\epsilon)} N(g(x))^r &= \sum_{x \in X} N(g(x))^r - \sum_{x \in A(\epsilon)} N(g(x))^r \\
&< t^r - (t^r - 2\epsilon) = 2\epsilon.
\end{aligned}$$

This permits us to estimate

$$(18.7) \quad \|f - g\|_r^r = \sum_{x \in A(\epsilon)} N(f(x) - g(x))^r + \sum_{x \in X \setminus A(\epsilon)} N(f(x) - g(x))^r$$

in terms of how close g is to f on $A(\epsilon)$ under these conditions, using (18.3) and (18.6).

Suppose for the moment that $V = k = \mathbf{Q}$, equipped with the restriction of the standard absolute value function on \mathbf{R} to \mathbf{Q} . Using the remarks in the previous paragraph, one can show that the sphere of radius t in $\ell^r(X, \mathbf{Q})$ centered at 0 has topological dimension 0. Of course, the same argument shows every sphere in $\ell^r(X, \mathbf{Q})$ has topological dimension 0, which means that $\ell^r(X, \mathbf{Q})$ has topological dimension ≤ 1 . This is basically the same as Example III 5 on p25f of [8], in which one takes $X = \mathbf{Z}_+$, $r = 2$, and $t < 1$. The argument in [8] uses an embedding of the sphere into the Hilbert cube, but this is just a convenience. Basically the same type of argument can be used for the sphere as for the Hilbert cube, because of the remarks in the previous paragraph. This is the other part of Erdős' famous theorem that $\ell^2(\mathbf{Z}_+, \mathbf{Q})$ has topological dimension equal to 1.

Now let $|\cdot|$ be an ultrametric absolute value function on any field k , and let N be an ultranorm on V with respect to $|\cdot|$ on k . In this case, one can use the earlier remarks to show that spheres in $\ell^r(X, V)$ with respect to $\|f\|_r$ are strongly 0-dimensional. In particular, this implies that $\ell^r(X, V)$ has topological dimension ≤ 1 . If X has infinitely many elements, $|\cdot|$ is nontrivial on k , and $V \neq \{0\}$, then we have already seen that $\ell^r(X, V)$ does not have topological dimension 0, as in Section 16. It follows that $\ell^r(X, V)$ also has topological dimension 1 under these conditions.

19 Uniform conditions

Let M be a nonempty set with a q -metric $d(x, y)$ for some $q > 0$. Let us say that M is *uniformly totally separated* if for each $r > 0$ there is an $\eta(r) > 0$ such that for every $x, y \in M$ with $d(x, y) \geq r$ there are $\eta(r)$ -separated sets $U, V \subseteq M$ with $x \in U$, $y \in V$, and $U \cup V = M$. Note that this implies that M is strongly totally separated, as in Section 15. If M is uniformly totally separated, then it follows that

$$(19.1) \quad \text{for each } r > 0 \text{ there is an } \eta(r) > 0 \text{ such that for every } x, y \in M \\
\text{with } d(x, y) \geq r, \text{ we have that } x \text{ and } y \text{ cannot be connected by} \\
\text{an } \eta(r)\text{-chain in } M.$$

More precisely, if M is uniformly totally separated, then (19.1) holds with the same choice of $\eta(r)$ as in the initial definition.

Conversely, suppose that M satisfies (19.1), and let $r > 0$ and $x \in M$ be given. Put

$$(19.2) \quad U = \{z \in M : x \text{ can be connected to } z \text{ by an } \eta(r)\text{-chain in } M\},$$

where $\eta(r)$ is as in (19.1). Thus $x \in U$ automatically, and it is easy to see that $U, M \setminus U$ are $\eta(r)$ -separated in M . By hypothesis, $M \setminus U$ contains every $y \in M$ with $d(x, y) \geq r$, which is the same as saying that

$$(19.3) \quad U \subseteq B(x, r).$$

In particular, this implies that M is uniformly totally separated, with $V = M \setminus U$, and with the same choice of $\eta(r)$.

Let us say that M is *uniformly 0-dimensional* if for each $r > 0$ there is an $\eta(r) > 0$ such that for every $x \in M$ there is an open set $U \subseteq M$ with $x \in U$ that satisfies (19.3) and has the property that $U, M \setminus U$ are $\eta(r)$ -separated in M . This condition clearly implies that M is strongly 0-dimensional, and that M is uniformly totally separated, with the same choice of $\eta(r)$. In fact, the argument in the previous paragraph shows that uniformly totally separated spaces are uniformly 0-dimensional, with the same choice of $\eta(r)$. This is because $U \subseteq M$ is automatically an open set when $U, M \setminus U$ are separated in M .

As a variant of this, let us say that M is uniformly totally separated at $x \in M$ if for each $r > 0$ there is an $\eta(x, r) > 0$ such that for every $y \in M$ with $d(x, y) \geq r$ there are $\eta(x, r)$ -separated sets $U, V \subseteq M$ with $x \in U, y \in V$, and $U \cap V = \emptyset$. This implies that

$$(19.4) \quad \begin{aligned} &\text{for each } r > 0 \text{ there is an } \eta(x, r) > 0 \text{ such that for every } y \in M \\ &\text{with } d(x, y) \geq r, \text{ we have that } x \text{ and } y \text{ cannot be connected by} \\ &\text{an } \eta(x, r)\text{-chain in } M, \end{aligned}$$

with the same choice of $\eta(x, r)$ as in the previous definition. Conversely, suppose that M satisfies (19.4), and let $r > 0$ be given. Put

$$(19.5) \quad U = \{z \in M : x \text{ can be connected to } z \text{ by an } \eta(x, r)\text{-chain in } M\},$$

where $\eta(x, r)$ is as in (19.4). This is the same as (19.2), but with $\eta(r)$ replaced by $\eta(x, r)$. As before, $x \in U$ automatically, and $U, M \setminus U$ are $\eta(x, r)$ -separated in M . Our hypothesis (19.4) says exactly that U also satisfies (19.3). This implies that M is uniformly totally separated at x , with $V = M \setminus U$, and with the same choice of $\eta(x, r)$.

Let us say that M is strongly 0-dimensional at $x \in M$ if for each $r > 0$ there is an $\eta = \eta(x, r) > 0$ and an open set $U \subseteq M$ such that $x \in U$, U satisfies (19.3), and $U, M \setminus U$ are η -separated in M . Thus M is strongly 0-dimensional as defined in Section 15 if and only if M is strongly 0-dimensional at each point $x \in M$. If M is strongly 0-dimensional at x , then it is easy to see that M is

uniformly totally separated at x , with the same choice of $\eta(x, r)$. Conversely, if M is strongly totally separated at x , then M is strongly 0-dimensional at x , with the same choice of $\eta(x, r)$, by the argument in the previous paragraph. As usual, this uses the fact that $U \subseteq M$ is an open set when $U, M \setminus U$ are separated in M .

Suppose that $K \subseteq M$ is compact, and that M is strongly 0-dimensional at each $x \in K$. Let $r > 0$ be given, so that for each $x \in K$ there is an $\eta(x, r) > 0$ and an open set $U(x, r)$ such that $x \in U(x, r)$,

$$(19.6) \quad U(x, r) \subseteq B(x, r),$$

and

$$(19.7) \quad U(x, r), M \setminus U(x, r) \text{ are } \eta(x, r)\text{-separated in } M.$$

Because K is compact, there are finitely many points $x_1, \dots, x_n \in K$ such that

$$(19.8) \quad K \subseteq \bigcup_{j=1}^n U(x_j, r).$$

Put

$$(19.9) \quad \eta = \min_{1 \leq j \leq n} \eta(x_j, r) > 0,$$

so that

$$(19.10) \quad U(x_j, r), M \setminus U(x_j, r) \text{ are } \eta\text{-separated in } M$$

for each $j = 1, \dots, n$, by (19.7). Let $w \in K$ be given, and let j be an integer such that $1 \leq j \leq n$ and $w \in U(x_j, r)$, as in (19.8). This implies that

$$(19.11) \quad d(x_j, w) < r,$$

by (19.6) with $x = x_j$, and hence that

$$(19.12) \quad B(x_j, r) \subseteq B(w, 2^{1/q} r),$$

since $d(\cdot, \cdot)$ is a q -metric on M . It follows that

$$(19.13) \quad U(x_j, r) \subseteq B(w, 2^{1/q} r),$$

by combining (19.6) with $x = x_j$ and (19.12). This shows that M satisfies a version of being strongly 0-dimensional at each $w \in K$, with a choice of $\eta > 0$ that depends on the radius and not w . In particular, if M is compact and strongly 0-dimensional, then M is uniformly 0-dimensional.

If $d(\cdot, \cdot)$ is an ultrametric on M , then M is uniformly 0-dimensional, with $\eta(r) = r$, for the same reasons as for strong 0-dimensionality in Section 15. If $d(\cdot, \cdot)$ is any q -metric on M , M is uniformly 0-dimensional, and $Y \subseteq M$, then it is easy to see that Y is also uniformly 0-dimensional with respect to the restriction of $d(\cdot, \cdot)$ to Y , and with the same choice of $\eta(r)$.

20 Some examples and remarks

Of course, a subset E of the real line is totally disconnected with respect to the standard topology on \mathbf{R} if and only if the interior of E is empty, which is the same as saying that $\mathbf{R} \setminus E$ is dense in \mathbf{R} . In this case, E has topological dimension 0, at least when $E \neq \emptyset$, if that is included in the definition.

Let E be a subset of \mathbf{R} again, and let x, y be distinct elements of E . We may as well suppose that $x < y$, since this can always be arranged by interchanging the roles of x and y , when needed. If $E \cap (x, y)$ is dense in (x, y) , then for each $\eta > 0$, x and y can be connected by an η -chain of elements of E with respect to the standard metric on \mathbf{R} . Thus if there is an $\eta > 0$ such that x and y cannot be connected by an η -chain of elements of E , then $E \cap (x, y)$ is not dense in (x, y) . This implies that

$$(20.1) \quad (x, y) \setminus \overline{E} \neq \emptyset.$$

Suppose now that $E \subseteq \mathbf{R}$ is strongly totally separated with respect to the restriction of the standard metric on \mathbf{R} to E . Under these conditions, the argument in the preceding paragraph implies that (20.1) holds for every $x, y \in E$ with $x < y$. The same conclusion holds when x and y are elements of the closure of $E \cap [x, y]$, by approximating x and y by elements of $E \cap [x, y]$, and applying the previous argument to those approximations. If x or y is not in the closure of $E \cap [x, y]$, then it is easy to see that (20.1) still holds. It follows that (20.1) holds for every $x, y \in \mathbf{R}$ with $x < y$ when E is strongly totally separated in \mathbf{R} , which means that $\mathbf{R} \setminus \overline{E}$ is dense in \mathbf{R} .

Conversely, if $\mathbf{R} \setminus \overline{E}$ is dense in \mathbf{R} , then it is easy to see that E is strongly 0-dimensional, which implies that E is strongly totally separated. As usual, one should also ask that $E \neq \emptyset$ in the first part of the preceding statement, if that is included in the definition of being strongly 0-dimensional. More precisely, \overline{E} is strongly 0-dimensional when $\mathbf{R} \setminus \overline{E}$ is dense in \mathbf{R} . If E is also bounded, then \overline{E} is compact, and hence \overline{E} is uniformly 0-dimensional, as in the preceding section. Otherwise, the same argument implies that bounded subsets of \overline{E} are uniformly 0-dimensional.

Let us consider some rather different examples in the plane, with respect to the standard Euclidean metric. Suppose that E_j is a finite subset of $[0, 1] \times \{1/j\}$ for each positive integer j , and put

$$(20.2) \quad E = \bigcup_{j=1}^{\infty} E_j.$$

It is easy to see that E is strongly 0-dimensional in this situation, with respect to the restriction of the standard Euclidean metric on \mathbf{R}^2 to E . However, we can choose the E_j 's so that

$$(20.3) \quad [0, 1] \times \{0\} \subseteq \overline{E},$$

where \overline{E} is the closure of E in \mathbf{R}^2 . In particular, this implies that \overline{E} is not totally disconnected.

Let M be a nonempty set with a q -metric $d(x, y)$ for some $q > 0$. If $A, B \subseteq M$ are η -separated for some $\eta > 0$, then it is easy to see that their closures $\overline{A}, \overline{B}$ are η -separated in M as well. Put $E = A \cup B$, so that

$$(20.4) \quad \overline{E} = \overline{A} \cup \overline{B}.$$

If $A, B \neq \emptyset$, then $\overline{A}, \overline{B} \neq \emptyset$, and hence \overline{E} is not connected in M .

Suppose now that E is any subset of M which is strongly totally separated with respect to the restriction of $d(x, y)$ to $x, y \in E$. This means that for every $x, y \in E$ with $x \neq y$ there are an $\eta > 0$ and η -separated sets $A, B \subseteq E$ such that $x \in A, y \in B$, and $A \cup B = E$. As in the previous paragraph, \overline{A} and \overline{B} are also η -separated in M and satisfy (20.4). Thus \overline{E} has a property analogous to being strongly totally separated, but which only applies to distinct elements of E , instead of \overline{E} .

Similarly, let us suppose that $E \subseteq M$ is strongly 0-dimensional with respect to the restriction of $d(x, y)$ to $x, y \in E$. This implies that for each $x \in E$ and $r > 0$ there are an $\eta > 0$ and η -separated sets $A, B \subseteq E$ such that $x \in A, A \cup B = E$, and A is contained in the open ball centered at x with radius r . It follows that \overline{A} and \overline{B} are η -separated subsets of M that satisfy (20.4), and that \overline{A} is contained in the closed ball in M centered at x with radius r . This shows that \overline{E} is strongly 0-dimensional at every element of E .

Let us return to the case where $M = \mathbf{R}^2$ with the standard Euclidean metric, and let E be as in (20.2). If z is any element of \mathbf{R}^2 , then it is easy to see that $E \cup \{z\}$ is strongly totally separated, with respect to the restriction of the standard Euclidean metric on \mathbf{R}^2 to $E \cup \{z\}$. More precisely, $E \cup \{z\}$ is strongly 0-dimensional at every element of E , for essentially the same reasons as before. If $z \notin [0, 1] \times \{0\}$, then $E \cup \{z\}$ is strongly 0-dimensional at z too, for essentially the same reasons again. Otherwise, if $z \in [0, 1] \times \{0\}$, and if we choose the E_j 's so that (20.3) holds, then $E \cup \{z\}$ is not strongly 0-dimensional at z . In this case, for each $\eta > 0$, there is an η -chain of elements of $E \cup \{z\}$ that starts at z and can go a distance which is at least almost $1/2$. If w, z are distinct elements of $[0, 1] \times \{0\}$, and if (20.3) holds, then $E \cup \{w, z\}$ is not strongly totally separated. This is because w and z can be connected by an η -chain of elements of $E \cup \{w, z\}$ for every $\eta > 0$.

21 Some additional remarks

Let M be a nonempty set with a q -metric $d(x, y)$ for some $q > 0$, and let E be a subset of M . As before, E is strongly totally separated with respect to the restriction of $d(\cdot, \cdot)$ to E if for every $x, y \in E$ with $x \neq y$ there are an $\eta > 0$ and η -separated sets $A, B \subseteq E$ such that $x \in A, y \in B$, and $A \cup B = E$. This implies that A, B are relatively open in E , and hence that there are $t_1, t_2 > 0$ such that

$$(21.1) \quad B(x, t_1) \cap E \subseteq A, \quad B(y, t_2) \cap E \subseteq B.$$

Here $B(w, t)$ denotes the open ball in M centered at $w \in M$ with radius $t > 0$ with respect to $d(\cdot, \cdot)$, as usual. Note that (21.1) holds with $t_1 = t_2 = \eta$, but in

some circumstances (21.1) may hold with larger values of t_1, t_2 as well.

As in the previous section, the closures $\overline{A}, \overline{B}$ of A, B are η -separated in M too, and satisfy (20.4). Observe that

$$(21.2) \quad B(x, t_1) \cap \overline{E} \subseteq \overline{A}, \quad B(y, t_2) \cap \overline{E} \subseteq \overline{B},$$

by (21.1). Let x', y' be distinct elements of \overline{E} , and suppose that x, y are distinct elements of E that are very close to x', y' , respectively. If

$$(21.3) \quad d(x, x') < t_1, \quad d(y, y') < t_2,$$

where t_1, t_2 are as in (21.1), then (21.2) implies that

$$(21.4) \quad x' \in \overline{A}, \quad y' \in \overline{B}.$$

One might like to try to use an argument like this to show that \overline{E} is strongly totally separated too, but the problem is that t_1, t_2 may depend on x and y , so that (21.3) does not hold.

If E is uniformly totally separated, as in Section 19, then one can take η to depend only on a positive lower bound for $d(x, y)$. In this case, the argument indicated in the previous paragraph can be used, and in fact \overline{E} is also uniformly totally separated. If $E \subseteq \mathbf{R}$ and $\mathbf{R} \setminus \overline{E}$ is dense in \mathbf{R} , then (21.1) holds with respect to the standard metric on \mathbf{R} for any $t_1, t_2 > 0$ such that

$$(21.5) \quad t_1 + t_2 < |x - y|.$$

Of course, we already know that \overline{E} is strongly 0-dimensional in this situation, and hence strongly totally separated.

Let M be any nonempty set with a q -metric $d(\cdot, \cdot)$ again. A subset E of M is strongly 0-dimensional with respect to the restriction of $d(\cdot, \cdot)$ to E if for each $x \in E$ and $r > 0$ there are an $\eta > 0$ and η -separated sets $A, B \subseteq E$ such that $x \in A$, $A \cup B = E$, and

$$(21.6) \quad A \subseteq B(x, r).$$

This implies that A, B are relatively open in E , and in particular that

$$(21.7) \quad B(x, t) \cap E \subseteq A$$

for some $t > 0$. More precisely, (21.7) holds with $t = \eta$, but it may also hold with larger values of t . As before, $\overline{A}, \overline{B}$ are η -separated in M , and satisfy (20.4). We also have that

$$(21.8) \quad B(x, t) \cap \overline{E} \subseteq \overline{A} \subseteq \overline{B}(x, r)$$

because of (21.6) and (21.7). Using the second inclusion in (21.8), we get that \overline{E} is strongly 0-dimensional at every $x \in E$, as in the previous section. One might like to show that \overline{E} is strongly 0-dimensional at a point $x' \in \overline{E}$ by approximating x' by $x \in E$, in such a way that

$$(21.9) \quad d(x, x') < t,$$

where t is as in (21.7). This does not always work, because t may depend on x . This does work when E is uniformly 0-dimensional, which is equivalent to E being uniformly totally separated, as in Section 19. This also works when $E \subseteq \mathbf{R}$ and $\mathbf{R} \setminus \overline{E}$ is dense in \mathbf{R} , in which case we already know that \overline{E} is strongly 0-dimensional with respect to the restriction of the standard metric on \mathbf{R} to \overline{E} .

22 Another perspective

Let M be a nonempty set with a q -metric $d(x, y)$ for some $q > 0$, and let E be a subset of M . Put

$$(22.1) \quad Z(r) = \{(x, y) \in E \times E : d(x, y) \geq r\}$$

for each $r > 0$, and let Z be a subset of $Z(r)$ for some $r > 0$. Let us say that E is *uniformly totally separated along Z* if there is an $\eta > 0$ such that for each $(x, y) \in Z$ there are η -separated sets $A, B \subseteq E$ such that $x \in A$, $y \in B$, and $A \cup B = E$. Of course, this implies that $d(x, y) \geq \eta$ for every $(x, y) \in Z$, so that $Z \subseteq Z(\eta)$. Note that E is uniformly totally separated with respect to the restriction of $d(\cdot, \cdot)$ to E , as in Section 19, if and only if E is uniformly totally separated along $Z(r)$ for each $r > 0$.

Suppose that E is strongly totally separated with respect to the restriction of $d(\cdot, \cdot)$ to E . If $Z \subseteq E \times E$ is compact with respect to the corresponding product topology on $E \times E$, and if $x \neq y$ for every $(x, y) \in Z$, then it is easy to see that $Z \subseteq Z(r)$ for some $r > 0$. Let us check that E is uniformly totally separated along Z under these conditions. Because E is strongly totally separated, for each $(x, y) \in Z$ there are an $\eta > 0$ and η -separated sets $A, B \subseteq E$ such that $x \in A$, $y \in B$, and $A \cup B = E$. Remember that A and B are relatively open subsets of E , so that $A \times B$ is relatively open in $E \times E$. Using the compactness of Z , we get that there are finitely many positive real numbers η_1, \dots, η_n and pairs of subsets $A_1, B_1, \dots, A_n, B_n$ of E such that A_j, B_j are η_j -separated and $A_j \cup B_j = E$ for each $j = 1, \dots, n$, and

$$(22.2) \quad Z \subseteq \bigcup_{j=1}^n A_j \times B_j.$$

It follows that E is uniformly totally separated along Z , with

$$(22.3) \quad \eta = \min(\eta_1, \dots, \eta_n).$$

More precisely, if $(x, y) \in Z$, then $(x, y) \in A_j \times B_j$ for some j , and A_j, B_j satisfy the requirements for (x, y) needed to verify that E is uniformly totally separated along Z .

Let x', y' be distinct elements of \overline{E} , and let $\{x_j\}_{j=1}^\infty, \{y_j\}_{j=1}^\infty$ be sequences of elements of E that converge to x', y' , respectively. We may as well suppose that $x_j \neq y_j$ for each j , and indeed that

$$(22.4) \quad d(x_j, y_j) \geq d(x', y')/2$$

for each j , since these conditions will hold for all but finitely many j anyway. Also let Z be the set of these pairs (x_j, y_j) . Suppose that E is uniformly totally separated along Z , and let η be as in the definition of that property. Thus for each positive integer j there are η -separated sets $A_j, B_j \subseteq E$ such that $x_j \in A_j$, $y_j \in B_j$, and $A_j \cup B_j = E$. As before, this implies that $\overline{A_j}$ and $\overline{B_j}$ are η -separated and satisfy

$$(22.5) \quad \overline{A_j} \cup \overline{B_j} = \overline{E}$$

for each j . We also have that

$$(22.6) \quad B(x_j, \eta) \cap E \subseteq A_j, \quad B(y_j, \eta) \cap E \subseteq B_j$$

for each j , as in (21.1), and hence

$$(22.7) \quad B(x_j, \eta) \cap \overline{E} \subseteq \overline{A_j}, \quad B(y_j, \eta) \cap \overline{E} \subseteq \overline{B_j}$$

for each j , as in (21.2). If j is sufficiently large so that

$$(22.8) \quad d(x', x_j), d(y', y_j) < \eta,$$

then (22.7) implies that $x' \in \overline{A_j}$, $y' \in \overline{B_j}$.

If $A, B \subseteq E$ are η -separated for some $\eta > 0$ and satisfy $A \cup B = E$, then E is automatically uniformly totally separated along $A \times B$, since one can use A, B in the definition of being uniformly totally separated along $A \times B$ for every $(x, y) \in A \times B$. Similarly, if $A, B \subseteq \overline{E}$ are η -separated for some $\eta > 0$ and satisfy $A \cup B = \overline{E}$, then \overline{E} is uniformly totally separated along $A \times B$. In this case, $A \cap E$ and $B \cap E$ are η -separated subsets of E whose union is equal to E , and E is uniformly totally separated along $(A \cap E) \times (B \cap E)$. If $x' \in A$ and $y' \in B$, then there are sequences $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of elements of $A \cap E$ and $B \cap E$ that converge to x' and y' , respectively. In particular, E is uniformly totally separated along the set Z of pairs (x_j, y_j) under these conditions.

Let us say that $E \subseteq M$ is *uniformly 0-dimensional along* $K \subseteq E$ if for each $r > 0$ there is an $\eta = \eta(K, r) > 0$ such that for every $x \in K$ there are η -separated sets $A, B \subseteq E$ with $x \in A$, $A \cup B = E$, and $A \subseteq B(x, r)$. Of course, this implies that E is strongly 0-dimensional at each point in K . Note that E is uniformly 0-dimensional with respect to the restriction of $d(\cdot, \cdot)$ to E , as in Section 19, if and only if E is uniformly 0-dimensional along $K = E$. If $K \subseteq E$ is compact, and E is strongly 0-dimensional at each element of K , then E is uniformly 0-dimensional along K . This was already shown in Section 19, with slightly different terminology and notation.

Let $x' \in \overline{E}$ be given, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to x' . Also let K be the subset of E consisting of the x_j 's, and suppose that E is uniformly 0-dimensional along K . Under these conditions, one can check that \overline{E} is strongly 0-dimensional at x' . This is analogous to the arguments used earlier in this and the previous section. In particular, if E is uniformly 0-dimensional with respect to the restriction of $d(\cdot, \cdot)$ to E , then essentially the same type of argument shows that \overline{E} is uniformly 0-dimensional as well.

In the other direction, suppose that \overline{E} is strongly 0-dimensional at a point $x' \in \overline{E}$. Thus for each $r > 0$ there an $\eta > 0$ and η -separated sets $A, B \subseteq \overline{E}$ such that $x' \in A$, $A \cup B = \overline{E}$, and

$$(22.9) \quad A \subseteq B(x', r).$$

It follows that

$$(22.10) \quad A \subseteq B(x, 2^{1/q} r)$$

for every $x \in B(x', r)$, by the q -metric version of the triangle inequality. Of course, $A \cap E$ and $B \cap E$ are η -separated subsets of E whose union is equal to E . Let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to x' , and let K be the subset of E consisting of the x_j 's, as before. If E is strongly 0-dimensional at x_j for each j , and \overline{E} is strongly 0-dimensional at x' , then it is easy to see that E is uniformly 0-dimensional along K . One can verify this directly, or using the fact that \overline{E} is strongly 0-dimensional at x_j for each j too, as in Section 20. This means that \overline{E} is strongly 0-dimensional at each point in $K \cup \{x'\}$, which is a compact set. Hence \overline{E} is uniformly 0-dimensional along $K \cup \{x'\}$, as before, which implies that E is uniformly 0-dimensional along K .

Part III

Simple functions

23 Basic notions

Let k be a field, and let X be a nonempty set. If E is a subset of X , then we let $\mathbf{1}_E(x)$ be the *characteristic* or *indicator function* associated to E on X , equal to 1 when $x \in E$ and to 0 when $x \in X \setminus E$. More precisely, $\mathbf{1}_E(x)$ is considered here as a k -valued function on X , so that 0 and 1 refer to the additive and multiplicative identity elements in k .

Let V be a vector space over k , and let f be a V -valued *simple function* on X , which is to say a function on X that takes only finitely many values in V . Thus f can be expressed as

$$(23.1) \quad f(x) = \sum_{j=1}^n v_j \mathbf{1}_{E_j}(x),$$

where v_1, \dots, v_n are the nonzero values of f , without repetitions, and

$$(23.2) \quad E_j = f^{-1}(\{v_j\}) = \{x \in X : f(x) = v_j\}$$

for each $j = 1, \dots, n$. Note that the E_j 's are nonempty and pairwise disjoint in this representation of f . Conversely, if v_1, \dots, v_n are finitely many vectors in V , and if E_1, \dots, E_n are finitely many subsets of X , then (23.1) defines a V -valued simple function on X . As usual, it is easy to reduce to the case where

the E_j 's are pairwise disjoint, using the various intersections of the E_j 's and their complements in X . One can also reduce to the case where the v_j 's are nonzero and distinct, by combining the E_j 's as needed. In particular, the space of V -valued simple functions on X is a vector space over k with respect to pointwise addition and scalar multiplication.

Let \mathcal{A} be an algebra of measurable subsets of X , so that \mathcal{A} is a collection of subsets of X that contains \emptyset , X and is closed under finite unions, finite intersections, and complementation. A V -valued simple function f on X is said to be *measurable* with respect to \mathcal{A} if

$$(23.3) \quad f^{-1}(\{v\}) \in \mathcal{A}$$

for each $v \in V$. Of course, $f^{-1}(\{v\})$ is the empty set for all but finitely many $v \in V$ when f is simple. If f is a measurable V -valued simple function on X , then f can be expressed as in (23.1), where v_1, \dots, v_n are the nonzero values of f on X , without repetitions, and (23.2) is in \mathcal{A} for each j . Conversely, if v_1, \dots, v_n are finitely many vectors in V , and if E_1, \dots, E_n are finitely many elements of \mathcal{A} , then (23.1) defines a measurable V -valued simple function on V . This is clear when the E_j 's are pairwise disjoint, and otherwise one can reduce to this case as in the preceding paragraph. It follows that the space of V -valued measurable simple functions on X is a linear subspace of the vector space of all V -valued simple functions on X . Let $S(X, V)$ be the space of V -valued measurable simple functions on X , which implicitly depends on the algebra \mathcal{A} too.

It is easy to see that the product of two k -valued simple functions on X is a k -valued simple function on X , which is measurable when the first two functions are measurable. One can also multiply a k -valued simple function on X and a V -valued simple function on X to get another V -valued simple function on X , which is measurable when the first two functions are measurable. Note that a real-valued simple function on X is nonnegative at every point in X if and only if it can be expressed as a linear combination of indicator functions with nonnegative coefficients. Similarly, a real-valued measurable simple function on X is nonnegative on X if and only if it can be expressed as a linear combination of indicator functions of measurable sets with nonnegative coefficients. As usual, one can add and multiply such expressions, to get another expression of the same type.

24 Finitely-additive nonnegative measures

Let \mathcal{A} be an algebra of measurable subsets of a nonempty set X again, and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . This means that μ is a nonnegative extended-real-valued function on \mathcal{A} which is finitely additive on pairwise-disjoint measurable sets and satisfies $\mu(\emptyset) = 0$. If f is a nonnegative

real-valued measurable simple function on X , then f can be expressed as

$$(24.1) \quad f(x) = \sum_{j=1}^n t_j \mathbf{1}_{E_j}(x)$$

for some nonnegative real numbers t_1, \dots, t_n and measurable subsets E_1, \dots, E_n of X . In this case, the integral of f over X with respect to μ is defined as a nonnegative extended real number by

$$(24.2) \quad \int_X f \, d\mu = \sum_{j=1}^n t_j \mu(E_j),$$

with the standard convention that $0 \cdot \infty = 0$. As usual, one can check that the value of the integral does not depend on the particular representation (24.1) of f . If a is a nonnegative real number, then $a f(x)$ is also a nonnegative real-valued measurable simple function on X , and

$$(24.3) \quad \int_X a f \, d\mu = a \int_X f \, d\mu,$$

using the convention $0 \cdot \infty = 0$ again, when necessary. Similarly, if g is another nonnegative real-valued measurable simple function on X , then

$$(24.4) \quad \int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

If $f(x) \leq g(x)$ for every $x \in X$, then

$$(24.5) \quad \int_X f \, d\mu \leq \int_X g \, d\mu.$$

If f is a nonnegative real-valued measurable simple function on X , then $f(x)^r$ is a measurable simple function on X for each positive real number r , and we put

$$(24.6) \quad \|f\|_r = \|f\|_{L^r(X)} = \left(\int_X f(x)^r \, d\mu(x) \right)^{1/r}.$$

Also put

$$(24.7) \quad \|f\|_{L^\infty(X)} = \max\{t \geq 0 : \mu(f^{-1}(\{t\})) > 0\},$$

which may be described as the *essential maximum* of f on X . More precisely, the maximum on the right side of (24.7) is taken over all nonnegative real numbers t such that $\mu(f^{-1}(\{t\})) > 0$. Because f is a simple function, $f^{-1}(\{t\}) = \emptyset$ for all but finitely many t , so that $\mu(f^{-1}(\{t\})) = 0$ for all but finitely many t . Thus the right side of (24.7) reduces to the maximum of a finite set of nonnegative real numbers. This set is empty in the trivial situation where $\mu(X) = 0$, in which case we interpret (24.7) as being equal to 0. Equivalently, $\|f\|_{L^\infty(X)}$ is the smallest nonnegative real number such that

$$(24.8) \quad f(x) \leq \|f\|_{L^\infty(X)}$$

for almost every $x \in X$ with respect to μ .

Observe that

$$(24.9) \quad \|a f\|_{L^r(X)} = a \|f\|_{L^r(X)}$$

for every nonnegative real number a and $0 < r \leq \infty$, using the convention $0 \cdot \infty = 0$ again when needed. If g is another nonnegative real-valued measurable simple function on X , then

$$(24.10) \quad \|f + g\|_{L^r(X)} \leq \|f\|_{L^r(X)} + \|g\|_{L^r(X)}$$

when $1 \leq r \leq \infty$. This is version of Minkowski's inequality, which is straightforward when $r = 1$ and $r = \infty$. If $0 < r \leq 1$, then

$$(24.11) \quad (f(x) + g(x))^r \leq f(x)^r + g(x)^r$$

for every $x \in X$, as in (1.11), with $q_1 = r$ and $q_2 = 1$. This implies that

$$(24.12) \quad \|f + g\|_{L^r(X)}^r \leq \|f\|_{L^r(X)}^r + \|g\|_{L^r(X)}^r$$

when $0 < r \leq 1$, by integrating both sides of (24.11) over X with respect to μ .

Let us briefly consider the case where \mathcal{A} is the algebra of all subsets of a nonempty set X , and μ is *counting measure* on X . Thus $\mu(E)$ is defined to be the number of elements of $E \subseteq X$, which is a nonnegative integer when E has only finitely many elements, and which is interpreted as being $+\infty$ when E has infinitely many elements. If f is any nonnegative real-valued function on X , then f is automatically measurable on X , and the Lebesgue integral of f with respect to counting measure on X is the same as the sum

$$(24.13) \quad \sum_{x \in X} f(x),$$

defined as the supremum of the corresponding finite subsums, as in Section 9. If f is a nonnegative real-valued simple function on X , then this is consistent with (24.2). Note that (24.13) reduces to a finite sum when f has finite support in X , and that (24.13) is infinite when f is a nonnegative real-valued simple function on X whose support has infinitely many elements.

25 Vector-valued functions

Let \mathcal{A} be an algebra of subsets of a nonempty set X again, and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . Also let $|\cdot|$ be a q -absolute value function on a field k for some $q > 0$, and let N be a q -norm on a vector space V over k with respect to $|\cdot|$ on k . If $f(x)$ is a V -valued measurable simple function on X , then $N(f(x))$ is a nonnegative real-valued measurable simple function on X . Thus we put

$$(25.1) \quad \|f\|_r = \|f\|_{L^r(X, V)} = \left(\int_X N(f(x))^r d\mu(x) \right)^{1/r}$$

for every positive real number r , which is defined as an extended real number, as in the previous section. Similarly, we take $\|f\|_{L^\infty(X,V)}$ to be the essential maximum of $N(f(x))$ on X , as in (24.7). This is the same as saying that $\|f\|_{L^\infty(X,V)}$ is the smallest nonnegative real number such that

$$(25.2) \quad N(f(x)) \leq \|f\|_{L^\infty(X,V)}$$

for almost every $x \in X$ with respect to μ . Note that

$$(25.3) \quad \|a f\|_{L^r(X,V)} = |a| \|f\|_{L^r(X,V)}$$

for every $a \in k$ and $0 < r \leq \infty$, with the usual convention that $0 \cdot \infty = 0$.

If $g(x)$ is another V -valued measurable simple function on X , then we have that

$$(25.4) \quad N(f(x) + g(x))^r \leq (N(f(x))^q + N(g(x))^q)^{r/q}$$

for every $x \in X$ and positive real number r , by the q -norm version of the triangle inequality for N on V . If $r \leq q$, then it follows that

$$(25.5) \quad N(f(x) + g(x))^r \leq N(f(x))^r + N(g(x))^r$$

for every $x \in X$, as in (24.11), with r replaced by r/q . This implies that

$$(25.6) \quad \|f + g\|_{L^r(X,V)}^r \leq \|f\|_{L^r(X,V)}^r + \|g\|_{L^r(X,V)}^r$$

when $0 < r \leq q$, by integrating both sides of (10.4) with respect to μ on X . If $q \leq r < \infty$, then we get that

$$(25.7) \quad \|f + g\|_{L^r(X,V)}^q \leq \|f\|_{L^r(X,V)}^q + \|g\|_{L^r(X,V)}^q,$$

using (25.4) and (24.10), with r replaced by r/q in the latter. It is easy to check directly that (25.7) holds when $r = \infty$, using (25.4) with $r = q$. If N is an ultranorm on V , then (25.5) holds for every positive real number r , which implies that (25.6) holds when $0 < r < \infty$, as before. This corresponds to $q = \infty$, in which case we also have that

$$(25.8) \quad \|f + g\|_{L^\infty(X,V)} \leq \max(\|f\|_{L^\infty(X,V)}, \|g\|_{L^\infty(X,V)}),$$

as one can easily verify.

Note that

$$(25.9) \quad \{x \in X : f(x) \neq 0\} = X \setminus f^{-1}(\{0\})$$

is a measurable subset of X when f is a V -valued measurable simple function on X . If r is any positive real number, then

$$(25.10) \quad \|f\|_{L^r(X,V)} < \infty$$

if and only if

$$(25.11) \quad \mu(\{x \in X : f(x) \neq 0\}) < \infty.$$

Let $S_0(X, V)$ be the space of V -valued measurable simple functions on X that satisfy (25.11), which is a linear subspace of the space $S(X, V)$ of all V -valued measurable simple functions on X . Of course, (25.10) holds for every V -valued measurable simple function on X when $r = \infty$. Similarly, if $0 < r \leq \infty$, then

$$(25.12) \quad \|f\|_{L^r(X, V)} = 0$$

if and only if

$$(25.13) \quad \mu(\{x \in X : f(x) \neq 0\}) = 0.$$

Let us say that V -valued measurable simple functions f, g on X are *equivalent* when

$$(25.14) \quad \mu(\{x \in X : f(x) \neq g(x)\}) = 0,$$

in which case

$$(25.15) \quad \|f\|_{L^r(X, V)} = \|g\|_{L^r(X, V)}$$

for every $0 < r \leq \infty$. This defines an equivalence relation on $S(X, V)$, and we let $\tilde{S}(X, V)$ be the corresponding space of equivalence classes. This is the same as taking the quotient of $S(X, V)$ by the linear subspace consisting of functions equal to 0 almost everywhere on X with respect to μ . In particular, $\tilde{S}(X, V)$ is also a vector space over k in a natural way, and it is easy to see that $\|f\|_{L^\infty(X, V)}$ determines a well-defined q -norm on $\tilde{S}(X, V)$. Let $\tilde{S}_0(X, V)$ be the image of $S_0(X, V)$ in $\tilde{S}(X, V)$ under this quotient mapping, which consists of equivalence classes of V -valued measurable simple functions on X that satisfy (25.11). Thus $\tilde{S}_0(X, V)$ is a linear subspace of $\tilde{S}(X, V)$, since $S_0(X, V)$ is a linear subspace of $S(X, V)$. One can check that $\|f\|_{L^r(X, V)}$ determines a well-defined q -norm on $\tilde{S}_0(X, V)$ when $q \leq r < \infty$, and that $\|f\|_{L^r(X, V)}$ determines a well-defined r -norm on $\tilde{S}_0(X, V)$ when $0 < r \leq q$, by (25.3), (25.6), and (25.7).

Suppose for the moment that \mathcal{A} is the algebra of all subsets of a nonempty set X , and that μ is counting measure on X . In this case, every V -valued simple function on X is automatically measurable, and $S_0(X, V)$ is the same as the space $c_{00}(X, V)$ of V -valued functions on X with finite support. Any two functions on X that are equal almost everywhere with respect to counting measure are in fact equal everywhere on X , so that $\tilde{S}(X, V)$ is the same as $S(X, V)$, and $\tilde{S}_0(X, V)$ is the same as $S_0(X, V) = c_{00}(X, V)$. If f is a V -valued simple function on X , then $\|f\|_{L^\infty(X, V)}$ is the same as the supremum norm of f , as in Section 8. If r is a positive real number and $f \in S_0(X, V)$, then $\|f\|_{L^r(X, V)}$ is the same as $\|f\|_{\ell^r(X, V)}$, as in Section 10. If f is a V -valued simple function on X not in $S_0(X, V)$, then $\|f\|_{L^r(X, V)} = +\infty$, but $\|f\|_{\ell^r(X, V)}$ was not defined in Section 10, strictly speaking. However, if $\|f\|_{\ell^r(X, V)}$ were defined in the same way as in Section 10, then it would also be infinite under these conditions.

26 The unit interval

Let us consider the case where X is the closed unit interval $[0, 1]$, and \mathcal{A} is an algebra of subsets of $[0, 1]$ that includes all closed subintervals of $[0, 1]$. In

particular, a subset of $[0, 1]$ with only one element is considered as a closed interval of length 0, and hence should be in \mathcal{A} . Also let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) such that $\mu([0, 1]) < \infty$,

$$(26.1) \quad \lim_{t \rightarrow a+} \mu([a, t]) = 0$$

when $0 \leq a < 1$, and

$$(26.2) \quad \lim_{t \rightarrow b-} \mu([t, b]) = 0$$

when $0 < b \leq 1$. This implies that

$$(26.3) \quad \mu(\{a\}) = 0$$

for every $a \in [0, 1]$, and in some situations (26.1) and (26.2) can be derived from (26.3). Of course, these conditions hold when

$$(26.4) \quad \mu([a, b]) = b - a$$

for $0 \leq a \leq b \leq 1$.

Let k be a field with a q -absolute value function for some $q > 0$, and V be a vector space over k with a q -norm N , as before. If f is a V -valued measurable simple function on $[0, 1]$, then put

$$(26.5) \quad f_t(x) = \mathbf{1}_{[0, t]}(x) f(x)$$

for every $t, x \in [0, 1]$, where $\mathbf{1}_{[0, t]}$ is considered as a k -valued indicator function on $[0, 1]$. Thus $f_t(x)$ is a V -valued measurable simple function on $[0, 1]$ as a function of x for each $t \in [0, 1]$. By construction, $f_1 = f$, and $f_0(x) = 0$ for every $x \in (0, 1]$. This means that $f_0 = 0$ almost everywhere on $[0, 1]$ with respect to μ , by (26.3) with $a = 0$, and one could also change the definitions slightly to get $f_0 = 0$ everywhere on X . If $0 \leq t_1 \leq t_2 \leq 1$, then

$$(26.6) \quad f_{t_2}(x) - f_{t_1}(x) = \mathbf{1}_{(t_1, t_2]}(x) f(x)$$

for every $x \in [0, 1]$, so that

$$(26.7) \quad \|f_{t_2} - f_{t_1}\|_{L^r([0, 1], V)} = \left(\int_{(t_1, t_2]} N(f(x))^r d\mu(x) \right)^{1/r}$$

for every positive real number r . In particular,

$$(26.8) \quad \|f_{t_2} - f_{t_1}\|_{L^r([0, 1], V)} \leq \mu((t_1, t_2])^{1/r} \|f\|_{L^\infty([0, 1], V)}$$

for every $0 \leq t_1 \leq t_2 \leq 1$ and $r > 0$.

As in the preceding section, we can identify V -valued measurable simple functions on $[0, 1]$ that are equal almost everywhere with respect to μ , to get a vector space $\tilde{S}([0, 1], V)$ over k . We have seen that $\|\cdot\|_{L^r([0, 1], V)}$ determines a q -norm on $\tilde{S}([0, 1], V)$ when $r \geq q$, and an r -norm on $\tilde{S}([0, 1], V)$ when $0 < r \leq q$.

This leads to a q -metric on $\tilde{S}([0, 1], V)$ when $r \geq q$, and to an r -metric on $\tilde{S}([0, 1], V)$ when $0 < r \leq q$, as usual. In both cases, we get a topology on $\tilde{S}([0, 1], V)$ corresponding to $\|\cdot\|_{L^r(X, V)}$. It follows from (26.1), (26.2), and (26.8) that

$$(26.9) \quad t \mapsto f_t$$

leads to a continuous mapping from $[0, 1]$ into $\tilde{S}([0, 1], V)$ with respect to the topology on $\tilde{S}([0, 1], V)$ corresponding to $\|\cdot\|_{L^r([0, 1], V)}$ when $0 < r < \infty$. This shows that $\tilde{S}([0, 1], V)$ is pathwise connected with respect to this topology when $0 < r < \infty$. Essentially the same argument shows that $\tilde{S}([0, 1], V)$ is contractible with respect to this topology when $0 < r < \infty$. If g is any other V -valued measurable simple function on $[0, 1]$, then one can consider families of the form $f_t + g$, in order to get contractibility centered at g instead of 0. Note that (26.9) is not normally continuous with respect to the topology on $\tilde{S}([0, 1], V)$ corresponding to $\|\cdot\|_{L^\infty([0, 1], V)}$.

Of course, one can get contractibility of vector spaces over the real or complex numbers using scalar multiplication. In particular, if $k = \mathbf{R}$ or \mathbf{C} equipped with the standard absolute value function, then one can use this to get contractibility of $\tilde{S}([0, 1], V)$ with respect to the topology corresponding to $\|\cdot\|_{L^\infty([0, 1], V)}$. Otherwise, if k is a field equipped with an ultrametric absolute value function, and N is an ultranorm on V , then $\|\cdot\|_{L^\infty([0, 1], V)}$ satisfies the ultrametric version of the triangle inequality, as in (25.8). This means that $\|\cdot\|_{L^\infty([0, 1], V)}$ determines an ultranorm on $\tilde{S}([0, 1], V)$, so that the $\tilde{S}([0, 1], V)$ is uniformly 0-dimensional with respect to the corresponding ultrametric, as in Section 19.

27 Pushing measures forward

Let X, Y be nonempty sets, and let \mathcal{A}, \mathcal{B} be algebras of subsets of X, Y , respectively. Suppose that a mapping $\phi : X \rightarrow Y$ is *measurable* in the sense that $\phi^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{B}$. If μ is a finitely-additive nonnegative measure on (X, \mathcal{A}) , then it is easy to see that

$$(27.1) \quad \nu(E) = \mu(\phi^{-1}(E))$$

defines a finitely-additive nonnegative measure on (Y, \mathcal{B}) . This is the measure on Y obtained by *pushing μ forward* using ϕ . If f is a nonnegative real-valued simple function on Y that is measurable with respect to \mathcal{B} , then one can check that $f \circ \phi$ is a nonnegative real-valued simple function on X that is measurable with respect to \mathcal{A} , and that

$$(27.2) \quad \int_X f \circ \phi \, d\mu = \int_Y f \, d\nu$$

under these conditions.

Let V be a vector space over a field k again. If f is a V -valued simple function on Y that is measurable with respect to \mathcal{B} , then it is easy to see that

$f \circ \phi$ is a V -valued simple function on X that is measurable with respect to \mathcal{A} . Suppose that $|\cdot|$ is a q -absolute value function on k for some $q > 0$, and that N is a q -norm on V with respect to $|\cdot|$ on k . If f is as before, then $N(f(x))$ is a nonnegative real-valued measurable simple function on Y , and $N(f(\phi(x)))$ is a nonnegative real-valued measurable simple function on X . Observe that

$$(27.3) \quad \|f \circ \phi\|_{L^r(X,V)} = \|f\|_{L^r(Y,V)}$$

for every $0 < r \leq \infty$, using (27.2) when $r < \infty$, and going back to the definition of the essential maximum when $r = \infty$.

Suppose for the moment that $Y = [0, 1]$, and that \mathcal{B} includes all closed subintervals of $[0, 1]$, as in the previous section. Suppose also that

$$(27.4) \quad \nu([0, 1]) = \mu(X) < \infty,$$

and that ν satisfies the analogues of (26.1) and (26.2) in this situation, and hence (26.3). Let f be a measurable V -valued simple function on X , and put

$$(27.5) \quad f_t(x) = \mathbf{1}_{[0,t]}(\phi(x)) f(x)$$

for every $x \in X$ and $0 \leq t \leq 1$, where $\mathbf{1}_{[0,t]}$ is the indicator function on $[0, 1]$ associated to $[0, t]$. Equivalently,

$$(27.6) \quad f_t(x) = \mathbf{1}_{\phi^{-1}([0,t])}(x) f(x)$$

for every $x \in X$ and $0 \leq t \leq 1$, where $\mathbf{1}_{\phi^{-1}([0,t])}$ is the indicator function on X associated to $\phi^{-1}([0, t])$. Thus $f_t(x)$ is a V -valued measurable simple function of x on X for every $t \in [0, 1]$, $f_1 = f$, and $f_0(x) = 0$ when $\phi(x) \neq 0$. By hypothesis, $\{0\}$ is a measurable subset of Y with respect to \mathcal{B} , so that $\phi^{-1}(\{0\})$ is a measurable subset of X with respect to \mathcal{A} , and

$$(27.7) \quad \mu(\phi^{-1}(\{0\})) = \nu(\{0\}) = 0.$$

This implies that $f_0 = 0$ almost everywhere on X with respect to μ , although one could again make some changes to get $f_0 = 0$ everywhere on X , if desired. If $0 \leq t_1 \leq t_2 \leq 1$, then we have that

$$(27.8) \quad f_{t_2}(x) - f_{t_1}(x) = \mathbf{1}_{(t_1,t_2]}(\phi(x)) f(x)$$

for every $x \in X$, and hence

$$(27.9) \quad \|f_{t_2} - f_{t_1}\|_{L^r(X,V)} = \left(\int_{\phi^{-1}((t_1,t_2])} N(f(x))^r d\mu(x) \right)^{1/r}$$

for every positive real number r . It follows that

$$(27.10) \quad \begin{aligned} \|f_{t_2} - f_{t_1}\|_{L^r(X,V)} &\leq \mu(\phi^{-1}((t_1,t_2]))^{1/r} \|f\|_{L^\infty(X,V)} \\ &= \nu((t_1,t_2])^{1/r} \|f\|_{L^\infty(X,V)} \end{aligned}$$

when $0 \leq t_1 \leq t_2 \leq 1$ and $0 < r < \infty$. As in Section 25, we can identify V -valued measurable simple functions on X that are equal almost everywhere with respect to μ to get a vector space $\tilde{S}(X, V)$ over k , and $\|\cdot\|_{L^r(X, V)}$ determines a q -norm on $\tilde{S}(X, V)$ when $r \geq q$, and an r -norm on $\tilde{S}(X, V)$ when $0 < r \leq q$. Using (27.10) and the analogues of (26.1) and (26.2) for ν , we get that

$$(27.11) \quad t \mapsto f_t$$

leads to a continuous mapping from $[0, 1]$ into $\tilde{S}(X, V)$ with respect to the topology on $\tilde{S}(X, V)$ corresponding to $\|\cdot\|_{L^r(X, V)}$ when $0 < r < \infty$. As before, this implies that $\tilde{S}(X, V)$ is pathwise connected with respect to this topology when $0 < r < \infty$, and in fact contractible. One can also consider $f_t + g$ for any other V -valued measurable simple function g on X , to get contractibility centered at g instead of 0.

Suppose now that X, Y are topological spaces, and that \mathcal{A}, \mathcal{B} are the σ -algebras of Borel subsets of X, Y , respectively. If ϕ is a continuous mapping from X into Y , then ϕ is automatically measurable with respect to the Borel sets. In particular, we can apply the discussion in the preceding paragraph to the case where $Y = [0, 1]$ with the standard topology. It is well known that there are continuous mappings ϕ from topological Cantor sets X onto $[0, 1]$, and that one can do this in such a way that Lebesgue measure on $[0, 1]$ corresponds to pushing forward a finite nonnegative Borel measure μ on X . Thus connectedness of X as a topological space is not really needed here.

28 Countability conditions

Let X be a nonempty set, let \mathcal{A} be an algebra of subsets of X , and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . Also let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let N be a q -norm on V with respect to $|\cdot|$ on k . Suppose for the moment that μ takes values in a set of finitely or countably many nonnegative extended real numbers, and that N takes values in a set of finitely or countably many nonnegative real numbers too. In particular, these conditions hold when \mathcal{A} has only finitely or countably many elements, and V has only finitely or countably many elements. This implies that $\|f\|_{L^\infty(X, V)}$ takes values in a set of finitely or countably many nonnegative real numbers when f is a V -valued measurable simple function on X . As in Section 25, we can identify V -valued measurable simple functions on X that are equal almost everywhere with respect to μ , to get a vector space $\tilde{S}(X, V)$ over k , and $\|\cdot\|_{L^\infty(X, V)}$ determines a q -norm on $\tilde{S}(X, V)$. Under the conditions just described, this q -norm takes values in a set of only finitely or countably many nonnegative real numbers, which implies that $\tilde{S}(X, V)$ has topological dimension 0 with respect to topology determined by the corresponding q -metric. Of course, if N is an ultranorm on V , then $\|\cdot\|_{L^\infty(X, V)}$ determines an ultranorm on $\tilde{S}(X, V)$, and one does not need any countability conditions to get that $\tilde{S}(X, V)$ has topological dimension 0.

If μ and N take values in sets with only finitely or countably many values again, then for each positive real number r , $\|f\|_{L^r(X,V)}$ takes values in a set of only finitely or countably many nonnegative extended real numbers when f is a V -valued measurable simple function on X . Let us now restrict our attention to V -valued measurable simple functions f on X that satisfy (25.11), and hence (25.10), which corresponds to the linear subspace $S_0(X, V)$ of $S(X, V)$. This leads to a linear subspace $\tilde{S}_0(X, V)$ of $\tilde{S}(X, V)$, after identifying such functions that are equal almost everywhere with respect to μ , as in Section 25. Remember that $\|\cdot\|_{L^r(X,V)}$ determines a q -norm on $\tilde{S}_0(X, V)$ when $r \geq q$, and an r -norm when $0 < r \leq q$. If μ and N take values in sets with only finitely or countably many elements, then $\tilde{S}_0(X, V)$ has topological dimension 0 with respect to the q or r -metric associated to $\|\cdot\|_{L^r(X,V)}$ for each $r > 0$, for the same reasons as before.

Suppose for the rest of the section that $|\cdot|$ is an ultrametric absolute value function on k , and that N is an ultranorm on V with respect to $|\cdot|$ on k . If N does not already take values in a set of finitely or countably many nonnegative real numbers, then we can basically reduce to that case by modifying N , as in Section 17. As before, let $h(t)$ be a monotonically increasing real-valued function defined on the set of nonnegative real numbers such that $h(0) = 0$ and $h(t) > 0$ when $t > 0$. Thus

$$(28.1) \quad h(N(v - w))$$

defines an ultrametric on V which determines the same topology on V as the ultrametric $N(v - w)$ associated to N . If f, g are V -valued measurable simple functions on X , then $h(N(f(x) - g(x)))^r$ is a nonnegative real-valued measurable simple function on X for every positive real number r , and we put

$$(28.2) \quad d_r(f, g) = \left(\int_X h(N(f(x) - g(x)))^r d\mu(x) \right)^{1/r}.$$

It is easy to see that (28.2) satisfies the r -metric version of the triangle inequality for each $r > 0$ under these conditions, for essentially the same reasons as for

$$(28.3) \quad \|f - g\|_{L^r(X,V)},$$

as in Section 25. If f, g also satisfy (25.11), which is to say that $f, g \in S_0(X, V)$, then $f - g \in S_0(X, V)$ too, which implies that (28.2) is finite. As usual, we get a vector space $\tilde{S}_0(X, V)$ by identifying V -valued measurable simple functions on X that satisfy (25.11) and which are equal almost everywhere with respect to μ , and (28.2), (28.3) define r -metrics on $\tilde{S}_0(X, V)$. If $h(t)$ and t are each bounded by positive constant multiples of the other on $[0, +\infty)$, then (28.2) and (28.3) are each bounded by the same constant multiples of the other on $S_0(X, V)$, and so the corresponding r -metrics on $\tilde{S}_0(X, V)$ are bounded by the same constant multiples of each other. This implies that the corresponding r -metrics determine the same topology on $\tilde{S}_0(X, V)$.

As in Section 17, we can choose $h(t)$ so that it takes values in a countable subset of \mathbf{R} , in addition to the properties already mentioned. If μ takes values in

a set of finitely or countably many nonnegative extended real numbers, then it follows that for each positive real number r , (28.2) takes values in a set of finitely or countably many nonnegative real numbers when $f, g \in S_0(X, V)$. This means that for each $0 < r < \infty$, the r -metric on $\tilde{S}_0(X, V)$ corresponding to (28.2) takes values in the same set of finitely or countably many nonnegative real numbers. Under these conditions, we get that $\tilde{S}_0(X, V)$ has topological dimension 0 with respect to the topology determined by the r -metrics corresponding to (28.2) or (28.3) when $0 < r < \infty$.

29 Measurable sets

The *symmetric difference* of two sets A, B is the set

$$(29.1) \quad A \triangle B = (A \setminus B) \cup (B \setminus A).$$

If C is another set, then

$$(29.2) \quad \begin{aligned} A \triangle C &= (A \setminus C) \cup (C \setminus A) \\ &\subseteq ((A \setminus B) \cup (B \setminus C)) \cup ((C \setminus B) \cup (B \setminus C)) \\ &= (A \triangle B) \cup (B \triangle C). \end{aligned}$$

Similarly, observe that

$$(29.3) \quad (A \cap C) \triangle (B \cap C) = (A \triangle B) \cap C$$

and

$$(29.4) \quad (A \cup C) \triangle (B \cup C) = (A \setminus C) \triangle (B \setminus C) = (A \triangle B) \setminus C.$$

Let X be a nonempty set, let \mathcal{A} be an algebra of subsets of X , and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . If $A, B \in \mathcal{A}$, then $A \triangle B \in \mathcal{A}$, and we put

$$(29.5) \quad d_\mu(A, B) = \mu(A \triangle B),$$

which is defined as a nonnegative extended real number. If $C \in \mathcal{A}$ too, then we get that

$$(29.6) \quad d_\mu(A, C) \leq d_\mu(A, B) + d_\mu(B, C),$$

by (29.2). Note that

$$(29.7) \quad d_\mu(A \cap C, B \cap C) \leq d_\mu(A, B)$$

and

$$(29.8) \quad d_\mu(A \cup C, B \cup C) \leq d_\mu(A, B),$$

by (29.3) and (29.4).

Let us say that $A, B \in \mathcal{A}$ are *equivalent* if (29.5) is equal to 0. This defines an equivalence relation on \mathcal{A} , and we let $\tilde{\mathcal{A}}$ denote the corresponding collection of equivalence classes. Put

$$(29.9) \quad \mathcal{A}_0 = \{A \in \mathcal{A} : \mu(A) < \infty\},$$

and observe that $A \triangle B \in \mathcal{A}_0$ when $A, B \in \mathcal{A}_0$, so that (29.5) is finite. Also let $\tilde{\mathcal{A}}_0$ be the subset of $\tilde{\mathcal{A}}$ consisting of equivalence classes of elements of \mathcal{A}_0 . It is easy to see that (29.5) leads to a well-defined metric on $\tilde{\mathcal{A}}_0$, by standard arguments.

Suppose for the moment that X , \mathcal{A} , and μ are as in Section 26, so that $X = [0, 1]$, \mathcal{A} contains all closed subintervals of $[0, 1]$, $\mu([0, 1]) < \infty$, and μ satisfies (26.1) and (26.2). In this case, one can use finite unions of subintervals of $[0, 1]$ to get families of elements of \mathcal{A} that depend on arbitrarily many parameters. More precisely, this leads to continuous families of elements of $\tilde{\mathcal{A}}$ with respect to the metric on $\tilde{\mathcal{A}}$ corresponding to (29.5), because of (26.1) and (26.2). Let us take μ to be as in (26.4), for simplicity. Under these conditions, it is easy to see that $\tilde{\mathcal{A}}$ has infinite topological dimension with respect to the topology determined by the metric corresponding to (29.5), using families of elements of \mathcal{A} like these.

Let X , \mathcal{A} , and μ be arbitrary again, let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k with a q -norm N with respect to $|\cdot|$ on k . If $A \subseteq X$, then let $\mathbf{1}_A^k(x)$ be the k -valued indicator function on X associated to A , so that $\mathbf{1}_A^{\mathbf{R}}$ is the real-valued indicator function. Let v_0 be an element of V , so that

$$(29.10) \quad \mathbf{1}_A^k(x) v_0$$

is a V -valued simple function on X , which is measurable when $A \in \mathcal{A}$. If $A, B \subseteq X$, then

$$(29.11) \quad \begin{aligned} N(\mathbf{1}_A^k(x) v_0 - \mathbf{1}_B^k(x) v_0) &= |\mathbf{1}_A^k(x) - \mathbf{1}_B^k(x)| N(v_0) \\ &= \mathbf{1}_{A \triangle B}^{\mathbf{R}}(x) N(v_0) \end{aligned}$$

for every $x \in X$. Note that (29.3) corresponds to multiplying this by $|\mathbf{1}_C^k(x)| = \mathbf{1}_C^{\mathbf{R}}(x)$. If $A, B \in \mathcal{A}$, then it follows that

$$(29.12) \quad \|\mathbf{1}_A^k v_0 - \mathbf{1}_B^k v_0\|_{L^r(X, V)} = \mu(A \triangle B)^{1/r} N(v_0)$$

for every positive real number r . Similarly,

$$(29.13) \quad \|\mathbf{1}_A^k v_0 - \mathbf{1}_B^k v_0\|_{L^\infty(X, V)} = \|\mathbf{1}_{A \triangle B}^{\mathbf{R}}\|_{L^\infty(X, \mathbf{R})} N(v_0)$$

is equal to $N(v_0)$ when $\mu(A \triangle B) > 0$, and to 0 otherwise.

30 Measurable sets, continued

Suppose for the moment that X , \mathcal{A} , and μ are as in Section 26 again, so that $X = [0, 1]$, \mathcal{A} contains all closed subintervals of $[0, 1]$, $\mu([0, 1]) < \infty$, and μ satisfies (26.1) and (26.2). Thus for each $x \in [0, 1]$ and $\epsilon > 0$ there is a $\delta(x) > 0$ such that

$$(30.1) \quad \mu([0, 1] \cap (x - \delta(x), x + \delta(x))) < \epsilon.$$

Because $[0, 1]$ is compact with respect to the standard topology, there are finitely many points x_1, \dots, x_n in $[0, 1]$ such that

$$(30.2) \quad [0, 1] \subseteq \bigcup_{j=1}^n (x_j - \delta(x_j)/2, x_j + \delta(x_j)/2).$$

If we put

$$(30.3) \quad \delta = \min_{1 \leq j \leq n} \delta(x_j)/2,$$

then we get that

$$(30.4) \quad \mu([0, 1] \cap (x - \delta, x + \delta)) < \epsilon$$

for every $x \in [0, 1]$. More precisely, for each $x \in [0, 1]$, we have that x is contained in one of the intervals on the right side of (30.2). This implies that

$$(30.5) \quad (x - \delta, x + \delta) \subseteq (x_j - \delta(x_j), x_j + \delta(x_j))$$

for some j , by the definition (30.3) of δ . Thus (30.4) follows from (30.5) and (30.1), where the latter is applied to x_j .

Now let X be a nonempty set, let \mathcal{A} be an algebra of subsets of X , and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . Suppose that for each $E \in \mathcal{A}$ with $\mu(E) < \infty$ and $\epsilon > 0$ there are finitely many measurable sets $A_1, \dots, A_n \subseteq X$ such that

$$(30.6) \quad \mu(A_j) < \epsilon$$

for each j , and

$$(30.7) \quad \bigcup_{j=1}^n A_j = E.$$

Put $E_0 = \emptyset$ and

$$(30.8) \quad E_l = \bigcup_{j=1}^l A_j$$

for $l = 1, \dots, n$, so that $E_{l-1} \subseteq E_l$ for $l = 1, \dots, n$ and $E_n = E$. If $d_\mu(\cdot, \cdot)$ is as in (29.5), then we have that

$$(30.9) \quad d_\mu(E_{l-1}, E_l) = \mu(E_l \setminus E_{l-1}) \leq \mu(A_l) < \epsilon$$

for $l = 1, \dots, n$, by (30.6). Let $\tilde{\mathcal{A}}_0$ be as in the previous section, so that $d_\mu(\cdot, \cdot)$ determines a metric on $\tilde{\mathcal{A}}_0$, as before. It follows from this discussion that $\tilde{\mathcal{A}}_0$ is chain connected with respect to this metric under these conditions.

Similarly, let k be a field with a q -absolute value function $|\cdot|$ for some $q > 0$, and let V be a vector space over k with a q -norm N . Also let $\tilde{S}_0(X, V)$ be as in Section 25, and remember that $\|\cdot\|_{L^r(X, V)}$ determines a q -norm on $\tilde{S}(X, V)$ when $q \leq r$, and an r -norm on $\tilde{S}_0(X, V)$ when $0 < r \leq q$. Under the conditions considered in the preceding paragraph, one can check that $\tilde{S}_0(X, V)$ is chain connected with respect to the q or r -metric corresponding to $\|\cdot\|_{L^r(X, V)}$ when

$0 < r < \infty$. More precisely, this uses the chain connectedness of $\tilde{\mathcal{A}}_0$ with respect to the metric corresponding to $d_\mu(\cdot, \cdot)$.

Let X be any nonempty set again, let \mathcal{A} be an algebra of subsets of X , and let μ be a finitely-additive nonnegative measure on (X, \mathcal{A}) . Suppose that $\tilde{\mathcal{A}}_0$ is chain connected with respect to the metric associated to $d_\mu(\cdot, \cdot)$, and let us show that we get the same type of conditions on X , \mathcal{A} , and μ as before, in (30.6) and (30.7). Let $E \subseteq X$ be a measurable set with $\mu(E) < \infty$, and let $\epsilon > 0$ be given. If $\tilde{\mathcal{A}}_0$ is chain connected, then there is an ϵ -chain in $\tilde{\mathcal{A}}_0$ that connects the elements of $\tilde{\mathcal{A}}_0$ corresponding to \emptyset and E . Equivalently, this means that there are finitely many measurable subsets E_0, E_1, \dots, E_n of X such that $E_0 = \emptyset$, $E_n = E$, $\mu(E_l) < \infty$ for each l , and

$$(30.10) \quad d_\mu(E_{l-1}, E_l) < \epsilon$$

for $l = 1, \dots, n$. If we put

$$(30.11) \quad E'_l = E_l \cap E$$

for each $l = 0, 1, \dots, n$, then we have that $E'_0 = \emptyset$, $E'_n = E$, and $E'_l \subseteq E$ for each l . We also have that

$$(30.12) \quad E'_l \setminus E'_{l-1} = (E_l \setminus E_{l-1}) \cap E \subseteq E_l \setminus E_{l-1}$$

for $l = 1, \dots, n$, and hence

$$(30.13) \quad \mu(E'_l \setminus E'_{l-1}) \leq \mu(E_l \setminus E_{l-1}) \leq d_\mu(E_{l-1}, E_l)$$

Similarly, put

$$(30.14) \quad E''_l = \bigcup_{j=1}^l E'_j$$

for $l = 1, \dots, n$, and $E''_0 = \emptyset$, so that $E''_{l-1} \subseteq E''_l$ for $l = 1, \dots, n$ by construction. Note that $E''_l \subseteq E$ for each l , because $E'_l \subseteq E$ for each l , and that $E''_n = E$. It is easy to see that

$$(30.15) \quad E''_l \setminus E''_{l-1} = E'_l \setminus E'_{l-1} \subseteq E'_l \setminus E'_{l-1}$$

for $l = 1, \dots, n$, which implies that

$$(30.16) \quad \mu(E''_l \setminus E''_{l-1}) \leq \mu(E'_l \setminus E'_{l-1}).$$

Put $A_l = E''_l \setminus E''_{l-1}$ for $l = 1, \dots, n$, so that

$$(30.17) \quad \mu(A_l) < \epsilon$$

for each l , by (30.10), (30.13), and (30.16). By construction, the A_l 's are pairwise-disjoint measurable subsets of X such that

$$(30.18) \quad E''_l = \bigcup_{j=1}^l A_j$$

for $l = 1, \dots, n$. In particular, (30.18) is equal to $E''_n = E$ when $l = n$, as desired.

31 Continuous simple functions

Let X be a nonempty topological space, and let Z be a nonempty set. A Z -valued function f on X is said to be *locally constant* at a point $x \in X$ if f is constant on an open subset of X that contains x . Similarly, f is said to be locally constant on X if f is locally constant at every element of X . It is easy to see that this happens if and only if $f^{-1}(\{z\})$ is an open set in X for every $z \in Z$. This implies that the inverse image of every subset of Z under f is an open set in X , since the union of any collection of open subsets of X is also an open set in X . It follows that the inverse image of every subset of Z under f is a closed set in X too, because its complement is an open set. In particular, $f^{-1}(\{z\})$ is a closed set in X for every $z \in Z$ when f is locally constant.

Of course, if f is locally constant on X , then f is continuous with respect to any topology on Z . If Z is equipped with the discrete topology, then every continuous mapping into Z is locally constant. If X is connected, then every locally constant function on X is constant. Conversely, if X is not connected, and if Z has at least two elements, then there is a locally constant mapping from X into Z which is not constant. Similarly, X is totally separated if and only if the collection of locally constant mappings from X into any set Z with at least two elements separates points in X .

Let \mathcal{A} be the collection of subsets of X that are both open and closed. It is easy to see that this defines an algebra of subsets of X . Also let k be a field, and let V be a vector space over k . In this case, a V -valued function f on X is locally constant on X if and only if

$$(31.1) \quad f^{-1}(\{v\}) \in \mathcal{A}$$

for every $v \in V$, by the earlier remarks. Thus a V -valued simple function f on X is measurable with respect to \mathcal{A} , as in Section 23, if and only if f is locally constant.

Let $CS(X, V)$ be the space of V -valued simple functions on X that are locally constant. This is the same as the space $S(X, V)$ defined in Section 23 when \mathcal{A} is as in the preceding paragraph. In particular, $CS(X, V)$ is a vector space over k with respect to pointwise addition and scalar multiplication, which can easily be verified directly as well. Every element of $CS(X, V)$ is continuous with respect to any topology on V , as before. If V is equipped with any topology that satisfies the first separation condition, and if a V -valued simple function f on X is continuous with respect to this topology on V , then one can check that f is locally constant on X .

If f is a locally constant function on X and $K \subseteq X$ is compact, then f takes only finitely many values on K . Suppose that f is a locally constant V -valued function on K , so that

$$(31.2) \quad \{x \in X : f(x) \neq 0\}$$

is a closed set in X , which is the same as the support of f in this case. If (31.2) is compact, then f takes only finitely many values on (31.2), and hence f takes only finitely many values on X too. Let $CS_{com}(X, V)$ be the collection of locally

constant V -valued functions on X with compact support. Thus $CS_{com}(X, V)$ is contained in $CS(X, V)$, by the previous remarks, and in fact $CS_{com}(X, V)$ is a linear subspace of $CS(X, V)$.

Let \mathcal{A}_{com} be the collection of subsets of X that are open, closed, and compact. Of course, compact subsets of X are automatically closed when X is Hausdorff, and closed sets in X are compact when X is compact. If X is any topological space, then $\emptyset \in \mathcal{A}_{com}$, and the union of any two elements of \mathcal{A} is an element of \mathcal{A}_{com} as well. Similarly, if $A \in \mathcal{A}_{com}$, and $B \subseteq X$ is both open and closed, then $A \cap B$ and $A \setminus B$ are elements of \mathcal{A}_{com} . If we put

$$(31.3) \quad \mathcal{A}_1 = \{A \subseteq X : A \in \mathcal{A}_{com} \text{ or } X \setminus A \in \mathcal{A}_{com}\},$$

then one can check that \mathcal{A}_1 is an algebra of subsets of X , which is obviously contained in the algebra \mathcal{A} defined earlier.

If $A \subseteq X$ is an open set and \mathcal{B} is a base for the topology of X , then A can be expressed as a union of elements of \mathcal{B} . If $A \subseteq X$ is compact and open, then it follows that A can be expressed as the union of finitely many elements of \mathcal{B} . If \mathcal{B} has only finitely or countably many elements, then there can only be finitely or countably many subsets of X that are compact and open. This implies that \mathcal{A}_{com} has only finitely or countably many elements, and hence that \mathcal{A}_1 has only finitely or countably many elements.

If X is compact, then $CS_{com}(X, V) = CS(X, V)$, and $\mathcal{A}_1 = \mathcal{A}$. Otherwise, suppose for the moment that X is not compact. Let $CS_1(X, V)$ be the collection of locally constant V -valued simple functions f on X such that

$$(31.4) \quad X \setminus f^{-1}(\{v_0\}) \text{ is a compact subset of } X$$

for some $v_0 \in V$. This is the same as saying that

$$(31.5) \quad f_0(x) = f(x) - v_0 \in CS_{com}(X, V),$$

so that $CS_1(X, V)$ is the same as the linear span in $CS(X, V)$ of $CS_{com}(X, V)$ and the space of V -valued constant functions on X . If f is an element of $CS_1(X, V)$, $v_0 \in V$ is as in (31.4), and $v \in V \setminus \{v_0\}$, then

$$(31.6) \quad f^{-1}(\{v\}) \subseteq X \setminus f^{-1}(\{v_0\}),$$

which implies that

$$(31.7) \quad f^{-1}(\{v\}) \text{ is a compact subset of } X.$$

In particular, (31.4) can hold for at most one element v_0 of V , because X is not compact. It follows from (31.4) and (31.7) that every element of $CS_1(X, V)$ is measurable with respect to \mathcal{A}_1 .

Conversely, suppose that f is a V -valued simple function on X that is measurable with respect to \mathcal{A}_1 . In particular, this means that f is measurable with respect to \mathcal{A} , since $\mathcal{A}_1 \subseteq \mathcal{A}$, so that f is locally constant on X . If $v \in V$, then $f^{-1}(\{v\}) \in \mathcal{A}_1$, so that either $f^{-1}(\{v\})$ is compact, or its complement in

X is compact. Because f is a simple function on X , $f^{-1}(\{v\}) \neq \emptyset$ for only finitely many $v \in V$. If $f^{-1}(\{v\})$ is compact for every $v \in V$, then X can be expressed as the union of finitely many compact sets, which implies that X is compact. Thus $f^{-1}(\{v_0\})$ is not compact for at least one $v_0 \in V$ when X is not compact. This implies that $X \setminus f^{-1}(\{v_0\})$ is compact for at least one $v_0 \in V$, because $f^{-1}(\{v_0\}) \in \mathcal{A}_1$. It follows that f is an element of $CS_1(X, V)$, so that $CS_1(X, V)$ consists of exactly the V -valued simple functions on X that are measurable with respect to \mathcal{A}_1 when X is not compact.

Suppose now that X is a locally compact Hausdorff topological space with topological dimension 0. If $K \subseteq X$ is compact, $W \subseteq X$ is an open set, and $K \subseteq W$, then there is an open set $U \subseteq X$ such that $K \subseteq U$ and U is compact, as in Section 14. If X is not compact, then one can apply this with $W = X$ to get that the one-point compactification of X has topological dimension 0 too.

Let $|\cdot|$ be a q -absolute value function on k for some $q > 0$, and let N be a q -norm on V with respect to $|\cdot|$ on k . Also let f be a continuous V -valued function on X , with respect to the topology on V determined by the q -metric associated to N . If $U \subseteq X$ is compact and open, then one can approximate f uniformly on U by locally constant V -valued simple functions on U . More precisely, for each $x \in U$, there is an open set $U(x) \subseteq U$ such that $x \in U$ and f is almost constant on $U(x)$, because f is continuous at x . One can also choose $U(x)$ to be compact, because X has topological dimension 0 at x . It follows that there are finitely many elements x_1, \dots, x_n of U such that

$$(31.8) \quad U = \bigcup_{j=1}^n U(x_j),$$

since U is compact, and $U(x_j) \subseteq U$ for each j . If we put $U_1 = U(x_1)$ and

$$(31.9) \quad U_l = U(x_l) \setminus \left(\bigcup_{j=1}^{l-1} U(x_j) \right)$$

for $l = 2, \dots, n$, then U_1, \dots, U_n are pairwise-disjoint compact open subsets of U such that

$$(31.10) \quad U = \bigcup_{l=1}^n U_l.$$

By construction, $U_l \subseteq U(x_l)$ for each l , which implies that f is approximately constant on U_l for each l . Thus one can approximate f by a V -valued function that is constant on U_l for each l .

If f has compact support in X , then we can first choose a compact open set $U \subseteq X$ that contains the support of f . The preceding argument permits us to approximate f uniformly on X by elements of $CS_{com}(X, V)$ with support contained in U . Similarly, if f vanishes at infinity on X , then there is a compact set $K \subseteq X$ such that f is small on $X \setminus K$, and we can choose U so that $K \subseteq U$. In this case, the preceding argument implies that $CS_{com}(X, V)$ is dense in $C_0(X, V)$ with respect to the supremum norm.

32 Lengths of chains

Let M be a nonempty set, and let $d(x, y)$ be a q -metric on M for some $q > 0$. If

$$(32.1) \quad x_1, \dots, x_n$$

is a finite sequence of elements of M and a is a positive real number, then let us define the a -length of (32.1) to be

$$(32.2) \quad \left(\sum_{j=1}^{n-1} d(x_j, x_{j+1})^a \right)^{1/a},$$

which is interpreted as being equal to 0 when $n = 1$. The analogue of (32.2) with $a = \infty$ is

$$(32.3) \quad \max_{1 \leq j < n} d(x_j, x_{j+1}),$$

which should also be interpreted as being equal to 0 when $n = 0$. As in (9.8) and (9.11), (32.3) is less than or equal to (32.2) for every $a > 0$, and (32.2) is monotonically decreasing in a . Note that (32.3) is less than some $\eta > 0$ exactly when (32.1) is an η -chain in M , as in Section 15.

If $1 \leq l \leq m \leq n$, then

$$(32.4) \quad d(x_l, x_m)^q \leq \sum_{j=l}^{m-1} d(x_j, x_{j+1})^q \leq \sum_{j=1}^n d(x_j, x_{j+1})^q,$$

by the q -metric version of the triangle inequality. This implies that

$$(32.5) \quad \max_{1 \leq l \leq m \leq n} d(x_l, x_m) \leq \left(\sum_{j=1}^{n-1} d(x_j, x_{j+1})^q \right)^{1/q},$$

so that the diameter of (32.1) in M is less than or equal to its q -length. Similarly, if $d(\cdot, \cdot)$ is an ultrametric on M , then

$$(32.6) \quad \max_{1 \leq l \leq m \leq n} d(x_l, x_m) \leq \max_{1 \leq j < n} d(x_j, x_{j+1}),$$

which is the analogue of (32.5) with $q = \infty$.

If $0 < a < b < \infty$, then

$$(32.7) \quad \sum_{j=1}^{n-1} d(x_j, x_{j+1})^b \leq \left(\max_{1 \leq j < n} d(x_j, x_{j+1}) \right)^{b-a} \sum_{j=1}^{n-1} d(x_j, x_{j+1})^a.$$

Equivalently, this means that

$$(32.8) \quad \left(\sum_{j=1}^{n-1} d(x_j, x_{j+1})^b \right)^{1/b} \leq \left(\max_{1 \leq j < n} d(x_j, x_{j+1}) \right)^{1-(a/b)} \left(\left(\sum_{j=1}^{n-1} d(x_j, x_{j+1})^a \right)^{1/a} \right)^{a/b}.$$

In particular, if the a -length (32.2) of (32.1) is bounded, and if the maximal step size (32.3) is small, then (32.8) says that the b -length of (32.1) should be small too when $a < b$.

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